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EXISTENCE OF SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS IN R^n

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Our purpose is to find results similar to those of Noussair and Swanson[1] for R^n instead of exterior domains. We extend our earlier results[2] to the case in which the nonlinear term depends on the first derivatives of u, too.

Let $|x|$ denote the Euclidean norm of a point $x = (x_1, x_2, \ldots, x_n)$ in real n-dimensional Euclidean space R^n. The Hölder norms of a function $u : \bar{M} \to R$ on the closure $\bar{M}$ of a bounded domain $M \subset R^n$ are defined by

$$\|u\|_{\alpha,\bar{M}} = \sup_{\substack{x,y\in\bar{M} \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}$$

$$\|u\|_{m+\alpha,\bar{M}} = \sum_{|i|=m} \|D^i u\|_{\alpha,\bar{M}} + \sup_{x\in\bar{M}} \sum_{|i|\leq m} |D^i u(x)|$$

$$D^i = \frac{\partial^{i_1+i_2+\cdots+i_n}}{\partial x_1^{i_1}\, \partial x_2^{i_2} \ldots \partial x_n^{i_n}}, \quad 0 < \alpha < 1, \; m = 1, 2, \ldots,$$

where i denotes a multi-index of length $|i| = i_1 + i_2 + \ldots + i_n$.

For convenience $\|u\|_{0,\bar{M}}$ is defined as $\sup_{x\in\bar{M}} |u(x)|$. As usual, $C^{m+\alpha}(\bar{M})$ denotes the space of all functions $u : \bar{M} \to R$ such that $\|u\|_{m+\alpha,\bar{M}}$ is finite, The boundary ∂M of M is said to belong to the class $C^{m+\alpha}$ whenever every $x \in \partial M$ has a neighbourhood N such that $\partial M \cap N$ can be represented in the form

$$x_l = h(x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n)$$

for some integer i, where $h \in C^{m+\alpha}$ on the closure of an appropriate bounded domain.

Consider the quasilinear elliptic differential equation

$$Lu \equiv -\sum_{i,j=1}^{n} a_{ij}(x) D_i D_j u = f(x, u, \nabla u), \qquad \ldots(1)$$

$$x \in R^n, D_i = \frac{\partial}{\partial x_i}, \nabla = (D_1, D_2, \ldots, D_n)$$

in R^n with the following assumptions :

(i) Each $a_{ij} \in C^{2+\alpha}(\bar{M})$ and the matrix $(a_{ij}(x))$ is uniformly positive definite in every bounded domain $M \subset R^n$ (uniform ellipticity condition);

(ii) There is a number α in (0, 1), fixed in the sequel, such that $f \in C^{\alpha}(\bar{M} \times \bar{J} \times \bar{N})$ for every bounded domain $M \subset R^n$, every bounded interval $J \subset R^1$ and every bounded domain $N \subset R^n$;

(iii) For every bounded subdomain M of R^n, there exists a nonnegative continuous function g_M such that

$$|f(x, u, p)| \leqslant g_M(|u|)[1 + |p|]^2, x \in \bar{M}, u \in R^1, p \in R^n$$

(Nagumo condition);

(iv) $$\left|\frac{\partial f(x, u, p)}{\partial p_k}\right|(1 + |p|) + \left|\frac{\partial f(x, u, p)}{\partial u}\right| \leqslant g_M(|u|)[1 + |p|]^2;$$

(v) $$\left|\frac{\partial a_{ij}(x)}{\partial x_k}\right|(1 + |p|)^2 + \left|\frac{\partial f(x, u, p)}{\partial x_k}\right| \leqslant P(|p|, |u|)[1 + |p|]^2$$

where $P(\tau, t)$ is a function that continuous on the set $\{\tau \geqslant 0, t \geqslant 0\}$, increases monotonically with respect to t and approaches zero uniformly with respect to t in $[0, t_1]$ as $\tau \to \infty$, where t_1 is an arbitrary postive number;

(vi) $$\left|\frac{\partial f(x, u, p)}{\partial u}\right| \leqslant P(|p|, |u|)[1 + |p|]^2.$$

A solution u of (1) in R^n is defined to be a function $u \in C^{2+\alpha}(\bar{M})$ for every bounded subdomain $M \subset R^n$ such that u satisfies (1) at every $x \in R^n$. Subsolutions of (1), i. e. functions u satisfying $Lu \leqslant f(x, u, \nabla u)$, and supersolutions are defined similarly.

The following theorem will be needed (Theorem 3.1, page (266) of Ladyzhenskaya and Ural'tseva book[3]):

Theorem 0—For a bounded domain $S \subset R^n$, let $u \in C^2(S)$ be a solution of (1) in S under the assumptions (i) — (vi). Suppose that R is a bounded domain with $\bar{R} \subset S$. Then there exists a constant K, independent of u, such that

$$\max_{x \in \overline{R}} \; | \nabla u(x) | \leqslant K$$

where K is a constant depends only on

$$\operatorname*{Sup}_{x \in \bar{S}} \; | u(x) |, n, g_M(1), P(|p|, 1), \text{and dist}(R, \partial S).$$

Here ∂S is the boundary of S.

Lemma 1—Let L be as in (1) and suppose that conditions (i) — (vi) are satisfied, $0 < \alpha < 1$. If there exist solutions v, w of $Lv \geqslant f(x, v, \nabla v)$, $Lw \leqslant f(x, w, \nabla w)$, respectively in R^n such that $w(x) \leqslant v(x)$ for all $x \in R^n$ then there exists a sequence of functions u_j in R^n with the following properties:

(A) $u_j \in C^{2+\alpha}(\overline{B(j)})$; $B(j) = \{x \in R^n; |x| < j\}$,

(B) $u_j(x) = v(x)$ if $x \in (R^n/B(j)) = \{x \in R^n; |x| \geqslant j\}$,

(C) $Lu_j(x) = f(x, u_j, \nabla u_j)$ in $B(j)$ and

(D) $w(x) \leqslant u_j(x) \leqslant v(x)$ if $x \in R^n$, $j = 1, 2, \ldots$.

PROOF : Since w is a subsolution and v is a supersolution of (1) with $w(x) \leqslant v(x)$ in $\overline{B(j)}$, a theorem of Nagumo (see ref. 1) guarantees the existence of a solution

$u = U_j \in C^{2+\alpha}(\overline{B(j)})$ of the boundary value problem :

$Lu = f(x, u, \nabla u)$ in $B(j)$,

$u(x) = v(x)$ on $S_j = \{x \in R^n; |x| = j\}$

satisfying $w(x) \leqslant U_j(x) \leqslant v(x)$ for all $x \in \overline{B(j)}$. The extension $u_j(x)$ of $U_j(x)$ to R^n defined by $u_j(x) = v(x)$ for $|x| > j$ then has all the required properties (A), (B), (C), and (D) of Lemma 1.

Lemma 2—Let (u_j) be the sequence in Lemma 1 and suppose that conditions (i) — (vi) are satisfied. Let i be a positive integer and M be an arbitrary bounded domain such that $\overline{M} \subset B(i)$. Then there exists a positive constant K_0, depending on α, n, i, v and w but not on j, such that

$$\|u_j\|_{2+\alpha, \overline{M}} \leqslant K_0 \quad \text{for all } j \geqslant 1. \qquad \ldots(2)$$

PROOF : Let Q and R be bounded domains such that $\overline{M} \subset Q$,

$\overline{Q} \subset R$, $\bar{R} \subset B(i)$, $\partial Q \in C^{2+\alpha}$, and $\partial R \in C^{2+\alpha}$.

For $j \geqslant i$, property (C) of Lemma 1 shows that u_j satisfies $Lu_j = f(x, u_j, \nabla u_j)$ in $B(i)$. By Theorem 0, there exists a constant K independent of u_j such that

$$\max_{x \in \bar{R}} \; | \nabla u_j (x) | \leqslant C, \quad C \leqslant K \sup_{x \in B(i)} \; | u_j (x) | .$$

Property (D) of Lemma 1 then implies that $\{\nabla u_j\}$ as well as $\{u_j\}$ is uniformly bounded on $\bar{R}$.

Let $v_j (x), j \geqslant 1$, be the unique solution of the linear boundary value problem :

$$Lv_j = f(x, u_j (x), \nabla u_j (x)), \; x \in R,$$

$$v_j (x) = 0 \qquad x \in \partial R. \qquad \ldots(3)$$

Define $f_j (x) = f(x, u_j (x), \nabla u_j (x))$, which is uniformly bounded sequence on $\bar{R}$ on account of the uniform boundedness of $\{u_j\}$ and $\{\nabla u_j\}$. For any number $p > 1$ it follows that there exists a positive constant K_1, independent of j, such that

$$\| f_j \|_{L^p (R)} \leqslant K_1 \text{ for all } j \geqslant i.$$

The norm of the solution $v_j (x)$ of (3) in the Sobolev space $W_p^2 (R)$ therefore satisfies

$$\| v_j \|_{W_p^2 (R)} \leqslant K_2$$

for some constant K_2, independent of j, by L^p estimates of Agmon *et al.*[4]. With the choice $p = \dfrac{n}{1 - \alpha}$ the Sobolev embedding lemma[2] shows that $v_j \in C^{1+\alpha} (\bar{R})$ and

$$\| v_j \|_{1+\alpha, \bar{R}} \leqslant K_3 \| v_j \|_{W_p^2 (R)} \leqslant K_3 K_2, \; j \geqslant i \qquad \ldots(4)$$

for another positive constant independent of j.

Let $w_j (x)$ be the unique solution of the boundary value problem :

$$Lw (x) = 0, \; x \in R ;$$

$$w (x) = u_j (x) , \; x \in \partial R, j \geqslant i. \qquad \ldots(5)$$

Since $u_j (x) \leqslant v (x)$ for all $x \in \bar{R}$ and for all $j \geqslant i$ by Lemma 1, the maximum principle for elliptic equations shows that there exists a constant K_4 independent of j so that

$\|w_j\|_{0,\bar{R}} \leqslant K_4$ for all $j \geqslant i$. By classical interior Schauder estimates for (5), then the following inequalities hold :

$$\|wj\|_{2+\alpha,Q} \leqslant K_5 \|w_j\|_{0,\bar{R}} \leqslant K_5 K_4 \qquad \ldots(6)$$

where the constant K_5 is again independent of j.

It follows from (3) and (5) that $u = v_j + w_j$ is solution of the linear boundary value problem

$$Lu = f(x, u_j(x), \nabla u_j(x)), \; x \in R; \qquad \ldots(7)$$

$$u(x) = u_j(x) \;, \; x \in \partial R.$$

However, $u = u_j$ also is a solution of (7) by Lemma 1 and hence $u_j = v_j + w$ in $\bar{R}$ by the standard uniqueness theorem for (7).

It is then a consequence of (4) and (6) that

$$\|u_j\|_{1+\alpha,\bar{Q}} \leqslant K_6 \text{ for } j \geqslant i.$$

In view of the regularity assumption (ii) the function $f_j(x) = (f(x, u_j(x), \nabla u_j(x))$ in (7) satisfies $\|f_j\|_{\alpha,\bar{Q}} \leqslant K_7$ for all $j \geqslant i$, where K_7 is again independent of j. Since $u_j(x)$ is a solution of $Lu(x) = f_j(x)$ for $x \in \bar{Q}$ by (7), the interior Schauder estimate

$$\|u_j\|_{2+\alpha,\bar{M}} \leqslant K_8 [\|u_j\|_{0,\bar{Q}} + \|f_j\|_{\alpha,\bar{Q}}]$$

implies the conclusion (2) of Lemma 2.

Theorem 3—Under the hypothesis of Lemma 1, (1) has a solution u in R^n such that $w(x) \leqslant u(x) \leqslant v(x)$ through R^n.

PROOF : Let $\{u_j\}$ be the sequence in Lemma 1. Let $\bar{M}_i \subset B(i)$ and $M_{i+1} \supset M_i$, then by Lemma 2, there exists a positive constant K_0, independent of j, such that

$$\|u\|_{2+\alpha,M_i} \leqslant K_0 \text{ for all } j \geqslant i.$$

From Ascoli-Arzela theorem, the compactness of the injection

$$C^{2+\alpha}(\bar{M}_i) \to C^2(\bar{M}_i)$$

then implies that $\{u_j; j \geqslant 1\}$ has a subsequence $\left\{ u_j^1 \right\}$ which converges uniformly in

the $C^2(\bar{M}_l)$ norm to a function u^l on $\bar{M}_l$. Define $u_j^0 = u_j$ for convenience and define $\{u_j^l\}$ inductively to be a subsequence of $\{u_j^{l-1}\}$ which converges uniformly in the $C^2(\bar{M}_l)$ norm to a function u^l on $\bar{M}_l$, $i = 1, 2, \ldots$. Define u in R^n by $u(x) = u^i(x)$ if $x \in \bar{M}_l$, the definition is consistent since $M_{l+1} \supset M_l$ and $u^{i+1} = u^i$ on $\bar{M}_l$ obviously for each $i = 1, 2, \ldots$.

We shall show that the required solution of (1) in R^n is given by

$$u(x) = u^i(x) \quad \text{if } x \in M_l,\ i = 1, 2, \ldots$$

First, note that the diagonal sequence $\{u_j^l(x)\}$ converges to $u(x)$ for all $x \in R^n$. For any bounded domain $\bar{M} \subset R^n$, $\bar{M} \subset \bar{M}_l$ for some integer i, and hence $\{u_j^j ; j \geqslant i\}$ being a subsequence of $\{u_j^l ; j \geqslant i\}$ converges uniformly in the $C^2(\bar{M})$ norm to ($u^i = u$ on $\bar{M}$. In particular $\{u_j^j\}$ and $\{Lu_j^j\}$ converge uniformly on $\bar{M}$ to u and Lu, respectively. Since

$u_j^j(x) = f(x, u_j^j(x), \nabla u_j^j(x))$ by Lemma 1, it follows that $u(x)$ is a solution of (1) in R^n of class $C^2(\bar{M})$, and hence of class $C^{2+\alpha}(\bar{M})$ by standard regularity argument based on Schauder estimates. Since $w(x) \leqslant u_j^j(x) \leqslant v(x)$ for each $j = 1, 2, \ldots$, the function u also satisfies $w(x) \leqslant u(x) \leqslant v(x)$ for all $x \in R^n$.

Corollary 4—Let L be as in (1). Suppose, in addition to the other assumptions bellow (1), that $f(x, 0, 0) \geqslant 0$ for all $x \in R^n$. Then a necessary and sufficient condition for the existence of a nonnegative solution of (1) in R^n. is the existence of a nonnegative supersolution of (1) in R^n. This follows easily by the choice $w(x) \equiv 0$ in Theorem 3.

Corollary 5— Suppose that the coefficients $a_{ij}(x) = a_{ij}$ in (1) are constants, that the conditions (i) - (vi) are fulfilled, $Lu = L(-iD)u$; $L(\xi) > 0$ for all $\xi \in R^n$ and there exist real numbers $\mu < 0$, $\xi_0 > 0$ such that $\xi f(., \xi, p) \leqslant \mu \xi^2$ for every ξ and p with $|\xi| \geqslant \xi_0$.

Then eqn. (1) has at least one solution $u \in C^{2+\alpha}(\bar{M})$.

PROOF : From the assumptions follows the existence of a positive constant γ such that for any $\xi \in R$, $p \in R^n$

$$\epsilon f(., \xi, p) \leqslant \gamma + \epsilon \mu \xi$$

where $\epsilon = \operatorname{sign} \xi$.

The linear boundary value problem

$$L_1 u = Lu - \mu u = \gamma \text{ in } R^n$$

has exactly one solution $\hat{v} \in C^{2+\alpha}(\bar{M})$, $\hat{v} \geqslant 0$; because of[2]

$$L_1(\xi) = L(\xi) - \mu > 0.$$

Hence $\bar{v} = -\hat{v}$ is the only solution of

$$L_1 u = Lu - \mu u = -\gamma \text{ in } R^n.$$

Then $L\hat{v} - \mu \hat{v} = \gamma$,

i. e. $$L\hat{v} = \gamma + \mu \hat{v} = \gamma + (\operatorname{sign} \hat{v}) \mu \hat{v}$$

$$L\hat{v} \geqslant \operatorname{sign}(\hat{v}).\ f(x, \hat{v}, \nabla \hat{v}) = f(x, \hat{v}, \nabla \hat{v}).$$

Similarly,

i. e. $$L\bar{v} = -\gamma + \mu \bar{v} = -\gamma - \operatorname{sign}(\bar{v})\, \mu \bar{v}$$

therefore

$$L\bar{v} \leqslant -\operatorname{sign}(\bar{v}).\ f(x, \bar{v}, \nabla \bar{v}) = f(x, \bar{v}, \nabla \bar{v}).$$

Now, since $\bar{v}$, $\hat{v}$ are sub-and supersolutions of (1) and $\bar{v} \leqslant \hat{v}$ then the assertion follows from Theorem 3.

References

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