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ON THE POINTS OF DISCONTINUITY OF A REAL VALUED FUNCTION

I. RAMABHADRASARMA

Department of Mathematics, Nagarjuna University, Nagarjunanagar 522510, *A. P.*

AND

S. KATRAGADDA

Department of Mathematics, V.R.S. and Y.S.N. College, Chirala, A. P.

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A n.s. c. for a set A of a topological space having a set P which is dense along with its complement, to be the set of points of discontinuity of a real valued function is obtained and a few applications are mentioned.

It is known that the set of points of discontinuity of a real valued function on any topological space is a F_σ set. In this note we prove a partial converse of this result and mention a few applications.

Theorem—If X is a topological space and has a set P in X such that P and its complement P^c are both dense in X, then a set A in X is the set of points of discontinuity of a real valued function on X if and only if A is a F_σ set.

PROOF : Let A be a F_σ set and $A = \bigcup_{n=1} F_n$ where $\{F_n\}$ is an expanding sequence of closed sets. Define

$$f_n(x) = \begin{cases} 0 \text{ if } x \in F_n^c \\ 1 \text{ if } x \in F_n \cap P \\ -1 \text{ if } x \in F_n \cap P^c \end{cases},$$

and

$$f(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}.$$

Clearly f is continuous on A^c. On the other hand if $x \in A$ and n is the smallest positive integer such that $x \in F_n$ then $f(x) = \frac{1}{2^{n-1}}$.

Choosing a net $\{x_\alpha\}$ in P^c that converges to x we get, $f_k(x_\alpha) = -1$ for all k and this establishes the discontinuity of f at x. That f is discontinuous at every point of P^c can be proved in a similar way.

Corollary 2—A n.s.c. for a subset A of R to be the set of points of discontinuity of a real valued function or R is that A is a F_σ set.

Remarks : If X is any set with at least two points and is endowed with discrete topology, then X has no proper dense sets and every function on X is continuous. Thus the existence of P as stated in Theorem 1 cannot be dropped.

Theorem 1 is not useful in the context of approximate continuity. Approximately continuous functions are continuous functions with respect to the density topology d on any interval contained in R, since the only dense subsets in this topology are fully measured sets. However the density topology contains other topologies where our theorem can be applied. One such topology is the r-topology introduced by O'Malley[1]. The r-topology on an interval I is the coarsest topology with respect to which every approximately differentiable function is continuous. An ambevalent set is a subset of I which is both F_σ and G_δ w.r.t. the usual topology.

Proposition 4—A subset A of I is the set of points of r-continuity of a real valued function on I if and only if A is a countable intersection of d-open sets each of which contains an equally measured set which is a countable union of ambevalent sets.

PROOF : It is known that the ambevalent d-open sets form a base for the r-topology and r-density is equivalent to density with respect to the usual topology. Further a set A is r-open if and only if A is d-open and contains an equally measured set which is a countable union of ambevalent sets. The result now follows from these facts and Theorem 1.

Our theorem can also be applied to the a.e. topology which was also introduced by O' Malley[1]. A set A is a.e. open if and only if it is d-open and its measure is equal to that of its interior. The a.e. topology is coarser than the r-topology and hence we have the following

Proposition 5—A subset A of I is the set of points of continuity of a real valued function on I with respect to the a.e. topology if A is the coantable intersection of d-open sets each having the same measure as its interior.

We now introduce another topology which is weaker than d-topology

Proposition 6—The collection $\mathcal{T}$ of fully measured subsets of open sets in I (w.r.t. the usual topology) is a topology on I.

PROOF : Routine.

Definition 7—The Topology of fully measured subsets of the open sets in I is called the near topology.

Lemma 8—A subset A of I is a G_δ (F_σ) set in the near topology iff A is a measurable.

PROOF : If A is measurable, A contains a G_δ set G such that $m(A \setminus G) = 0$. Let $G = \bigcap G_n$ where each G_n is open. The sets $A_n = G_n \setminus (A \setminus G)$ are $\mathcal{T}$-open, and $A = \bigcap A_n$.

Lemma 9— A measurable set P is $\mathcal{T}$-dense iff for every open interval $J \subseteq I$, $m(J \cap P) > 0$.

PROOF : Suppose P is not $\mathcal{T}$-dense, Let H be an $\mathcal{T}$-open set contained in P^c and V, an open set such that V contains H and $m(V \setminus H) = 0$. Then for any $J \subset V$, $m(J \cap P) = 0$. Conversely suppose that for some open interval J, $m(J \setminus A) = 0$. Let V be the union of all such open intervals. By Lindeloff's theorem there is a countable subcollection $\{J_n\}$ of these intervals whose union is V. Then $0 \leqslant m(V \cap A) = m((\cup J_n) \cap A) \leqslant \Sigma m(J_n \cap A) = 0$. Hence $m(V \cap A) = 0$. If $H = V \cap A^c$ then H is a fully measured subset of V and hence H is $\mathcal{T}$-open which is not contained in A^c. Then A is not $\mathcal{T}$-dense.

Theorem 10—A measurable set of points of near continuity of a real valued function iff A is measurable.

PROOF : Follows from Lemmas 8, 9 and Theorem 1.

Remark : The near topology is strictly coarser than the d-topology. It can further be shown that the Dini's theorem holds good in a weaker form in the near topology while this is not the case with respect to the density topology. Even though the notion of near continuity differs from that of (approximate) continuity at a given point, one can show that near continuity and continuity are equivalant on an interval I.

REFERENCE

1. R.J. O' Malley, *Pacific J. Math.* (1977), 30-46.