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A REMARK ON PAPERS OF G. PÓLYA AND P. K. KAMTHAN

H. S. KASANA*

Mathematics Department, University of Roorkee, Roorkee 247667 *(U.P.)*

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A short and simple proof in the context of (p, q)-order and (p, q)-type etc. is given of a well known theorem by Pólya (*Math.* Z., **29** (1929), 549-640) for entire power series and later on extended to entire Dirichlet series by Kamthan (*Indian J. pure. appl. Math.*, **1** (1970), 325-29).

1. Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, $(s = \sigma + it,\ 0 \leqslant \lambda_1 < \lambda_n < \lambda_{n+1} \to \infty$ as $n \to \infty)$ be an entire Dirichlet series. Set $M(\sigma, f) = \max_{t \in R} |f(\sigma + it)|$. Recently Juneja *et al.*[2,3] have introduced the concept of index-pair (p, q), (p, q)-order and (p,q)-type of an entire Dirichlet series, for nonnegative integers $p \geqslant q + 1$.

Define a constant $L(p, q) : (0 < L(p, q) < \infty)$ as

$$L(p, q) = \lim_{n \to \infty} \sup \frac{\log^{[p-1]} \lambda_n}{\log^{[q+1]} |a_n|^{-1/\lambda_n}} \qquad \ldots(1.1)$$

and $\delta(p, q)$ such that

$$\delta(p, q) = \lim_{n \to \infty} \sup \frac{\log n}{\lambda_n \exp^{[q-1]} (\log^{[p-2]} \lambda_n)^{1/L(p,q)}} \qquad \ldots(1.2)$$

where $\log^{[p]} x$ stands for the pth iterate of $\log x$.

Let $f(s)$ be such that $\delta(p, q) = 0$. Then $f(s)$ is of (p, q)-order ρ if and only if $L(p, q) < \infty$; and

$$\rho \equiv \rho(p, q) = P_1(L(p, q)) \qquad \ldots(1.3)$$

where,

$$P_1(L(p, q)) = \begin{cases} L(p, q) & \text{if } q + 1 < p < \infty \\ 1 + L(p, q) & \text{if } p = q + 1 = 2 \\ \max.(1, L(p, q)) & \text{if } 3 < q + 1 = p < \infty \\ \infty & \text{if } p = q = \infty. \end{cases}$$

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For a given entire Dirichlet series $f(s)$ of (p, q)-order ρ $(b < \rho < \infty)$, set

$$V \equiv V(p, q) = \limsup_{n\to\infty} \frac{\log^{[p-2]}\lambda_n}{(\log^{[q]} \mid a_n \mid^{-1}/\lambda n)^{\rho-A}} \qquad \ldots(1.4)$$

where $b = 0$ if $p > q + 1$ and $b = 1$ if $p = q + 1$, and $A = 1$ for $(p, q) = (2, 1)$ and zero otherwise.

Let $f(s)$ be an entire Dirichlet series with index-pair (p, q). If $0 < V < \infty$, the function $f(s)$ is of (p, q)-order ρ $(b < \rho < \infty)$ and (p, q)-type τ if and only if $\tau = MV$, where V is given by (1.4) and

$$M \equiv M(p, q) = \begin{cases} \dfrac{(\rho - 1)^{\rho-1}}{\rho^{\rho}} & \text{if } (p, q) = (2, 1) \\ \dfrac{1}{\rho e} & \text{if } (p, q) = (2, 0) \\ 1 & \text{for all other index-pairs.} \end{cases} \qquad \ldots(1.5)$$

The concept of proximate order for an entire function represented by Dirichlet series was introduced by Balaguer[1] and later on, extensively studied by Kamthan[4,5]. We define this function for entire Dirichlet series with index-pair (p, q) as follows.

Definition —By a proximate order of an entire Dirichlet series of (p, q)-order ρ $(b < \rho < \infty)$, we mean a real valued continuous function $\rho(\sigma)$ if it satisfies :

(i) $\rho(\sigma) \to \rho$ as $\sigma \to \infty$ and

(ii) $\Lambda^{1}_{[q]}(\sigma)\,\rho'(\sigma) \to 0$ as $\sigma \to \infty$, where $\rho'(\sigma)$ can be interpreted as either $\rho(\sigma -)$ or $\rho(\sigma +)$ when these are unequal, and for convenience.

$$\Lambda_{[q]}(\sigma) = \prod_{i=0}^{q} \log^{[i]} x.$$

Further, if

$$\limsup_{\sigma\to\infty} \frac{\log^{[p-1]} M(\sigma, f)}{(\log^{[q-1]} \sigma)^{\rho(\sigma)}} = \tau^*(p, q) \equiv \tau^*, (0 \leqslant \tau^* \leqslant \infty). \qquad \ldots(1.6)$$

The function $\rho(\sigma)$ is called a proximate order of a given entire function $f(s)$ with index-pair (p, q) if the quantity τ^* is non zero finite. We shall term τ^* as generalized (p, q)- type of $f(s)$.

§ 2. The following result whose power series analogue was considered by Póyla[8] is due essentially to Kamthan[6].

Theorem A—If $f(s)$ is an entire function satisfying

$$\lim_{n\to\infty} \frac{\sup \lambda_n}{\inf n} = \frac{d}{D}, d < \infty \qquad \ldots(2.1)$$

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = h > 0 \qquad \ldots(2.2)$$

$$h\,d \leqslant 1 \qquad \ldots(2.3)$$

and having finite Ritt-order ρ and $\{b_n\}$ is a complex sequence with the condition

$$\lim_{n\to\infty} \frac{\log |\, b_n \,|}{\lambda_n} = 0 \qquad \ldots(2.4)$$

then, the function $g(s) = \sum_{n=1}^{\infty} a_n b_n \exp(s\lambda_n)$ is an entire Dirichlet series of order ρ, same type and has the same proximate order and proximate type as that of $f(s)$.

In the present note, we give an elementary proof of Kamthan's Theorem and extend it to entire Dirichlet series with (p, q)-growth. It is significant to mention that the conditions (2.2) and (2.3) have been dropped and we replace the candition (2.1) by a less restrictive condition (1.2). It is also obvious that (2.1) implies (1.2) for $\delta(p,q)=0$ which we shall need and proof, being trivial, is left to the reader. Now, we prove

Theorem B—Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire Dirichlet series with index-pair (p, q), (p, q)-order ρ $(b < \rho < \infty)$, $\delta(p, q) = 0$ and $\{b_n\}$ be a complex sequence satisfying (2.4). Then the Hadamard composition $g(s) = \sum_{n=1}^{\infty} a_n b_n \exp(s\lambda_n)$ is an entire Dirichlet series with the same (p, q)-order, (p, q)-type, proximate order and generalized (p, q)-type as that of $f(s)$.

PROOF : Since $\log |\, b_n \,| = o(\lambda_n)$, it follows that

$$\frac{\log |\, a_n b_n \,|}{\lambda_n} \to -\infty \text{ as } n \to \infty.$$

Thus $g(s)$ is an entire Dirichlet series.

The (p, q)-order of $f(s)$ is given by (1.3). To claim that $f(s)$ and $g(s)$ have some $g(p, q)$-order, it is sufficient to show that $L(b, q)$ is same for both entire functions. From (2.4), we have for any $\epsilon > 0$ and $n > n_0(\epsilon)$

$$-\epsilon\lambda_n < \log |\, b_n \,| < \epsilon\lambda_n. \qquad \ldots(2.5)$$

Also,

$$\frac{1}{\lambda_n}\log \mid a_n\, b_n \mid^{-1} = \frac{\log \mid a_n \mid^{-1}}{\lambda_n}\left\{1 + \frac{\log \mid b_n \mid^{-1}}{\log \mid a_n \mid^{-1}}\right\}. \qquad \ldots(2.6)$$

On using (2.5) in (2.6), we get for sufficiently large n

$$\frac{\log^{[q+1]} \mid a_n \mid^{-1/\lambda_n}}{\log^{[p-1]}\lambda_n} - o(1) < \frac{\log^{[q+1]} \mid a_n\, b \mid^{-1/\lambda_n}}{\log^{[p-1]}\lambda_n}$$

$$< \frac{\log^{[q+1]} \mid a_n \mid^{-1/\lambda_n}}{\log^{[p-1]}\lambda_n} + o(1).$$

On taking limits in above we find that $L(p, q)$ is same for both the entire Dirichlet series $f(s)$ and $g(s)$.

Let $\tau_1^*(p, q) \equiv \tau_1^*$ be the generalized (p, q)-type of $g(s)$. Kasana[7] has proved that

$$\limsup_{n\to\infty} \frac{F(\log_{[p-2]}\lambda_n)}{\log^{[q]} \mid a_n \mid^{-1/\lambda_n}} = \left(\frac{\tau^*}{M}\right)^{1/\rho-A} \qquad \ldots(2.7)$$

where A and M have the same meaning as in (1.4) and (1.5) respectively and $F(t)$ is a real valued function assumed to have unique solution for $t > t_0$ such that

$$t = (\log^{[q-1]}\sigma)^{(\sigma)-A} \Leftrightarrow F(t) = \log^{[q-1]}\sigma.$$

Since $F(t)$ is increasing, taking (2.5) and (2.6) into consideration we again have

$$\frac{\log^{[q]} \mid a_n \mid^{-1/\lambda_n}}{F(\log^{[p-2]}\lambda_n)} - o(1) < \frac{\log^{[q]} \mid a_n\, b_n \mid^{-1/\lambda_n}}{F(\log^{[p-2]}\lambda_n)}$$

$$< \frac{\log^{[q]} \mid a_n \mid^{-1/\lambda_n}}{F(\log_{[p-2]}\lambda_n)} + o(1)$$

Hence, (in view of (2.7)),

$$\left(\frac{M}{\tau^*}\right)^{1/\rho-A} = \left(\frac{M}{\tau_1^*}\right)^{1/\rho-A}$$

which implies $\tau = \tau_1^*$

Finally, if we define $\rho(\sigma) = \rho$ and $F(t) = t^{1/\rho-A}$ then τ^* and τ_1^* are nothing but simply (p, q)-types of $f(s)$ and $g(s)$ respectively. This completes the proof of the theorem.

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