A REMARK ON PAPERS OF G. PÓLYA AND P. K. KAMTHAN

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A short and simple proof in the context of \((p, q)\)-order and \((p, q)\)-type etc. is given of a well known theorem by Pólya (Math. Z., 29 (1929), 549-640) for entire power series and later on extended to entire Dirichlet series by Kamthan (Indian J. pure appl. Math., 1 (1970), 325-29).

1. Let \(f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)\), \((s = \sigma + it, 0 \leq \lambda_1 < \lambda_2 < \cdots \rightarrow \infty\) as \(n \rightarrow \infty\)) be an entire Dirichlet series. Set \(M(\sigma, f) = \max_{t \in R} |f(\sigma + it)|\). Recently Juneja et al. have introduced the concept of index-pair \((p, q)\), \((p, q)\)-order and \((p, q)\)-type of an entire Dirichlet series, for nonnegative integers \(p \geq q + 1\).

Define a constant \(L(p, q) : (0 < L(p, q) < \infty)\) as

\[
L(p, q) = \lim_{n \to \infty} \sup \frac{\log^{(p-1)} \lambda_n}{\log^{(q+1)} \mid a_n \mid^{-1/\lambda_n}} \quad \ldots (1.1)
\]

and \(\delta(p, q) such that

\[
\delta(p, q) = \lim_{n \to \infty} \sup \frac{\log n}{\lambda_n \exp^{(p-1)} (\log^{(q+1)} \lambda_n)^{1/L(p, q)}} \quad \ldots (1.2)
\]

where \(\log_k x \) stands for the \(k\)th iterate of \(\log x\).

Let \(f(s)\) be such that \(\delta(p, q) = 0\). Then \(f(s)\) is of \((p, q)\)-order \(p\) if and only if \(L(p, q) < \infty\) and

\[
p \equiv p(p, q) = P_1(L(p, q)) \quad \ldots (1.3)
\]

where,

\[
P_1(L(p, q)) = \begin{cases} 
L(p, q) & \text{if } q + 1 < p < \infty \\
1 + L(p, q) & \text{if } p = q + 1 = 2 \\
\max \{1, L(p, q) \} & \text{if } 3 < q + 1 = p < \infty \\
\infty & \text{if } p = q = \infty.
\end{cases}
\]

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For a given entire Dirichlet series \( f(s) \) of \((p, q)\)-order \( \rho \) \((b < \rho < \infty)\), set

\[
V \equiv V(p, q) = \limsup_{n \to \infty} \frac{\log^{(p-1)/2} \lambda_n}{\log(a_n) \log^{-1/2} n} \quad \text{...(1.4)}
\]

where \( b = 0 \) if \( p > q + 1 \) and \( b = 1 \) if \( p = q + 1 \), and \( A = 1 \) for \((p, q) = (2, 1)\) and zero otherwise.

Let \( f(s) \) be an entire Dirichlet series with index-pair \((p, q)\). If \( 0 < V < \infty \), the function \( f(s) \) is of \((p, q)\)-order \( \rho \) \((b < \rho < \infty)\) and \((p, q)\)-type \( \tau \) if and only if \( \tau = MV \), where \( V \) is given by (1.4) and

\[
M \equiv M(p, q) = \begin{cases} 
\frac{(p - 1)^{p-1}}{\rho^p} & \text{if } (p, q) = (2, 1) \\
\frac{1}{\rho} & \text{if } (p, q) = (2, 0) \\
1 & \text{for all other index-pairs.}
\end{cases} \quad \text{...(1.5)}
\]

The concept of proximate order for an entire function represented by Dirichlet series was introduced by Balaguer\(^1\) and later on, extensively studied by Kamthan\(^4,5\). We define this function for entire Dirichlet series with index-pair \((p, q)\) as follows.

**Definition** — By a proximate order of an entire Dirichlet series of \((p, q)\)-order \( \rho \) \((b < \rho < \infty)\), we mean a real valued continuous function \( \rho(\sigma) \) if it satisfies:

(i) \( \rho(\sigma) \to \rho \) as \( \sigma \to \infty \) and \( \frac{\rho'}{\rho} \)

(ii) \( \Delta_{\{q\}}(\sigma) \rho'(\sigma) \to 0 \) as \( \sigma \to \infty \), where \( \rho'(\sigma) \) can be interpreted as either \( \rho(\sigma-) \) or \( \rho(\sigma+) \) when these are unequal, and for convenience.

\[
\Delta_{\{q\}}(\sigma) = \prod_{i=0}^{q} \log^{(i)} x.
\]

Further, if

\[
\limsup_{\sigma \to \infty} \frac{\log^{(p-1)} M(\sigma, f)}{\log^{(q-1)} \rho(s)} = \tau^*(p, q) \equiv \tau^*, \quad (0 \ll \tau^* \ll \infty). \quad \text{...(1.6)}
\]

The function \( \rho(\sigma) \) is called a proximate order of a given entire function \( f(s) \) with index-pair \((p, q)\) if the quantity \( \tau^* \) is non zero finite. We shall term \( \tau^* \) as generalized \((p, q)\)-type of \( f(s) \).

§ 2. The following result whose power series analogue was considered by Pólya\(^8\) is due essentially to Kamthan\(^8\).
**Remark on Papers of G. Pólya and P. K. Kamthan**

**Theorem A**—If $f(s)$ is an entire function satisfying

$$\lim_{n \to \infty} \sup_{n} \frac{\lambda_n}{\inf_{n}} = \frac{d}{D}, \quad d < \infty \quad \ldots \text{(2.1)}$$

$$\lim_{n \to \infty} (\lambda_{n+1} - \lambda_n) = h > 0 \quad \ldots \text{(2.2)}$$

$$h d \leq 1 \quad \ldots \text{(2.3)}$$

and having finite Ritt-order $\rho$ and $\{b_n\}$ is a complex sequence with the condition

$$\lim_{n \to \infty} \frac{\log |b_n|}{\lambda_n} = 0 \quad \ldots \text{(2.4)}$$

then, the function $g(s) = \sum_{n=1}^{\infty} a_n b_n \exp(s\lambda_n)$ is an entire Dirichlet series of order $\rho$, same type and has the same proximate order and proximate type as that of $f(s)$.

In the present note, we give an elementary proof of Kamthan's Theorem and extend it to entire Dirichlet series with $(p, q)$-growth. It is significant to mention that the conditions (2.2) and (2.3) have been dropped and we replace the condition (2.1) by a less restrictive condition (1.2). It is also obvious that (2.1) implies (1.2) for $\delta(p,q)=0$ which we shall need and proof, being trivial, is left to the reader. Now, we prove

**Theorem B**—Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ be an entire Dirichlet series with index-pair $(p, q), (p, q)$-order $\rho (b < \rho < \infty)$, $\delta (p, q) = 0$ and $\{b_n\}$ be a complex sequence satisfying (2.4). Then the Hadamard composition $g(s) = \sum_{n=1}^{\infty} a_n b_n \exp(s\lambda_n)$ is an entire Dirichlet series with the same $(p, q)$-order, $(p, q)$-type, proximate order and generalized $(p, q)$-type as that of $f(s)$.

**Proof:** Since $\log |b_n| = o(\lambda_n)$, it follows that

$$\frac{\log |a_n b_n|}{\lambda_n} \rightarrow -\infty \quad \text{as} \quad n \to \infty.$$ 

Thus $g(s)$ is an entire Dirichlet series.

The $(p, q)$-order of $f(s)$ is given by (1.3). To claim that $f(s)$ and $g(s)$ have some $g(p, q)$-order, it is sufficient to show that $L(b, q)$ is same for both entire functions. From (2.4), we have for any $\epsilon > 0$ and $n > n_0(\epsilon)$

$$-\epsilon \lambda_n < \log |b_n| < \epsilon \lambda_n. \quad \ldots \text{(2.5)}$$
Also,
\[ \frac{1}{\lambda_n} \log |a_n b_n|^{-1} = \frac{\log |a_n|^{-1}}{\lambda_n} \left\{ 1 + \frac{\log |b_n|^{-1}}{\log |a_n|^{-1}} \right\}. \] ... (2.6)

On using (2.5) in (2.6), we get for sufficiently large \( n \)
\[ \frac{\log^{(q+1)} |a_n|^{-1/\lambda_n}}{\log^{(p-1)} \lambda_n} - o(1) < \frac{\log^{(q+1)} |a_n b_n|^{-1/\lambda_n}}{\log^{(p-1)} \lambda_n} < \frac{\log^{(q+1)} |a_n|^{-1/\lambda_n}}{\log^{(p-1)} \lambda_n} + o(1). \]

On taking limits in above we find that \( L(p, q) \) is same for both the entire Dirichlet series \( f(s) \) and \( g(s) \).

Let \( \tau^*_1 (p, q) \equiv \tau^*_1 \) be the generalized \( (p, q) \)-type of \( g(s) \). Kasana\(^7\) has proved that
\[ \limsup_{n \to \infty} \frac{F(\log^{(p-2)} \lambda_n)}{\log^{(q)} |a_n|^{-1/\lambda_n}} = \left( \frac{\tau^*}{M} \right)^{1/\rho^\lambda} \] ... (2.7)
where \( A \) and \( M \) have the same meaning as in (1.4) and (1.5) respectively and \( F(t) \) is a real valued function assumed to have unique solution for \( t > t_0 \) such that
\[ t = (\log^{(q-1)} \sigma)^{(p-1)} \iff F(t) = \log^{(q-1)} \sigma. \]

Since \( F(t) \) is increasing, taking (2.5) and (2.6) into consideration we again have
\[ \frac{\log^{(q)} |a_n|^{-1/\lambda_n}}{F(\log^{(p-2)} \lambda_n)} - o(1) < \frac{\log^{(q)} |a_n b_n|^{-1/\lambda_n}}{F(\log^{(p-2)} \lambda_n)} < \frac{\log^{(q)} |a_n|^{-1/\lambda_n}}{F(\log^{(p-2)} \lambda_n)} + o(1) \]
Hence, (in view of (2.7)),
\[ \left( \frac{M}{\tau^*} \right)^{1/\rho^\lambda} = \left( \frac{M}{\tau^*_1} \right)^{1/\rho^\lambda} \]
which implies \( \tau = \tau^*_1 \)

Finally, if we define \( \rho (\sigma) = \rho \) and \( F(t) = t^{1/\rho^\lambda} \) then \( \tau^* \) and \( \tau^*_1 \) are nothing but simply \( (p, q) \)-types of \( f(s) \) and \( g(s) \) respectively. This completes the proof of the theorem.
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REFERENCES