GROUP-THEORETIC ORIGINS OF CERTAIN GENERATING FUNCTIONS
OF THE MODIFIED LAGUERRE POLYNOMIALS

S. N. SINGH AND R. N. BALA

Department of Mathematics, Banaras Hindu University, Varanasi 221005

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In this paper L. Weisner's group theoretic method of obtaining generating
functions is utilized in the case of modified Laguerre polynomials \( L_{a,b,m,n}(x) \), by giving suitable interpretation to the index \( n \), in order to derive the
elements of Lie-algebra.

1. INTRODUCTION

Recently Goyal\(^1\) has defined the modified Laguerre polynomials of degree \( n \) as follows:

\[
L_{a,b,m,n}(x) = \frac{b^n (m)_n}{n!} \, _1F_1 (-n; m; a \, x/b).
\]

We note the following results for this polynomials set

\[
DL_{a,b,m,n}(x) = \frac{1}{b^x} \{ b \, (1 - m - n) \, L_{a,b,m,n-1}(x) + nL_{a,b,m,n}(x) \} \ldots (1.1)
\]

\[
DL_{a,b,m,n}(x) = \frac{1}{b^x} \, \{ [b \, (-m - n) + a \, x] \, L_{a,b,m,n}(x) \\
+ (n + 1) \, L_{a,b,m+n+1}(x) \} \ldots (1.2)
\]

where

\[
D \equiv \frac{d}{dx}
\]

and

\[
XD^a \, L_{a,b,m,n}(x) + (m - ax/b) \, DL_{a,b,m,n}(x) + \frac{a}{b} \, nL_{a,b,m,n}(x) = 0.
\ldots (1.3)
\]
The object of the present paper is to derive some generating functions, which are believed to be new of the modified Laguerre polynomials, by suitable interpreting $n$ with the help of Weisner's group theoretic method.

2. GROUP-THEORETIC DISCUSSION

In order to use Weisner's method we construct from (1.3) following partial differential equation by replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}, n$ by $y \frac{\partial}{\partial y}$ and $L_{a,b,m,n} (x)$ by $u, (x,y)$:

$$x \frac{\partial^2 u}{\partial x^2} + (m - a x/b) \frac{\partial u}{\partial x} + \frac{a}{b} y \frac{\partial u}{\partial y} = 0.$$ \hspace{1cm} \ldots(2.1)

Let $L$ represent the differential operator of (2.1) i. e.,

$$L = x \frac{\partial^2}{\partial x^2} + (m - a x/b) \frac{\partial}{\partial x} + \frac{a}{b} y \frac{\partial}{\partial y}.$$ \hspace{1cm} \ldots(2.2)

We now find two linear differential operators $B$ and $C$. Let

$$B = B_1 (x, y) \frac{\partial}{\partial x} + B_2 (x, y) \frac{\partial}{\partial y} + B_0 (x, y)$$

with the aid of (1.1), we have

$$B [L_{a,b,m,n} (x) y^n] = B_1 y^n x^{-1} [b (1 - m - n) L_{a,b,m,n-1} (x)]$$

$$+ nL_{a,b,m,n} (x)] + B_2 [n y^{n-1}] L_{a,b,m,n} (x)$$

$$+ B_0 L_{a,b,m,n} (x) y^n.$$ \hspace{1cm} \ldots(2.3)

In order to make the coefficient of $L_{a,b,m,n-1} (x) y^{n-1}$ independent of $x$ and $y$ we choose $B_1 = x y^{-1}$. Then (2.3) becomes

$$B [L_{a,b,m,n} (x) y^n] = b (1 - m - n) L_{a,b,m,n-1} (x) y^{n-1} + L_{a,b,m,n} (x) y^n$$

$$\times \{y^{-1} (n + B_0 n) + B_0 \}.$$ \hspace{1cm} \ldots(2.4)

In (2.4) we can make the coefficient of $L_{a,b,m,n} (x) y^n$ equal to zero by choosing $B_2 = -1$ and $B_0 = 0$.

Therefore

$$B = x y^{-1} \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$ \hspace{1cm} \ldots(2.5)

and

$$B [L_{a,b,m,n} (x) y^n] = b (1 - m - n) L_{a,b,m,n-1} (x) y^{n-1}.$$ \hspace{1cm} \ldots(2.6)
Similarly it is fairly easy to observe that

\[ C = bxy \frac{\partial}{\partial x} + by^2 \frac{\partial}{\partial y} + (b \; m - a \; x) \; y \] \hspace{1cm} \ldots \text{(2.7)}

and consequently

\[ C \left[ L_{a,b,m,n} (x) \; y^n \right] = (n + 1) \; L_{a,b,m,n+1} (x) \; y^{n+1}. \] \hspace{1cm} \ldots \text{(2.8)}

To find the group of operators, let us write \( A = y \; \partial / \partial y \).

Then we have the following commutator relations,

\[ [A, B] = -B, \; [A, C] = C, \; [B, C] = -2 \; bA - bm. \] \hspace{1cm} \ldots \text{(2.9)}

We would like to prove that these operators commute with \( L \) or \( \psi(x) \; L \), where \( \psi(x) \) be a function of \( x \) yet to be determined. We express \( L \) or \( \psi(x) \; L \) in terms of these operators we know that

\[ Lu = x \; \frac{\partial^2 u}{\partial x^2} + (m - a \; x/b) \; \frac{\partial u}{\partial x} + \frac{a}{b} \; y \; \frac{\partial u}{\partial y}. \]

Also in making the computation of \([B, C]\), we find that

\[ CBu = bx^2 \frac{\partial^2 u}{\partial x^2} - by^2 \frac{\partial^2 u}{\partial y^2} + (a \; x - bm) \left( -x \; \frac{\partial u}{\partial x} + y \; \frac{\partial u}{\partial y} \right). \]

Again if we take

\[ \psi(x) = bx, \] we have

\[ [bx \; L - CB] \; u = b \; y^2 \; \frac{\partial^2 u}{\partial y^2} + b \; m \; y \; \frac{\partial u}{\partial y} \]

but

\[ b \; A^2 \; u = b \; y \; \frac{\partial}{\partial y} \left( y \; \frac{\partial u}{\partial y} \right). \]

Therefore

\[ [bx \; L - CB] \; u = (b \; A^2 + (m - 1) \; bA) \; u \]

or equivalently,

\[ bxL \; u = (CB + b \; A^2 + (m - 1) \; bA) \; u. \] \hspace{1cm} \ldots \text{(2.10)}

It follows from the relations (2.9) and (2.10) and may be verified by direct calculation, that \( bxL \) is commutative with \( A, B, C \) and hence with \( R = r_1 \; A + r_2 \; B + r_3 \; C + r_4 \) where \( r \)'s are arbitrary constants, \( R \) the set of linear differential operators,
The commutator relation (2.9) shows that the operator $1$, $A$, $B$, $C$ generates a Lie algebra. The element of this Lie algebra may be represented in the form $e^{k}$.

This Lie algebra determines a root system and a Weyl group. The extended form of the group generated by $B$ and $C$ is given by

$$e^{cC} f(x, y) = (1 - cby)^{-m} \exp \left( -\frac{a cxy}{1 - cby} \right) f \left( \frac{x}{1 - cby}, \frac{y}{1 - cby} \right) \ldots \quad (2.11)$$

$$e^{kB} f(x, y) = f \left( \frac{xy}{y - g}, y - g \right) \ldots \quad (2.12)$$

By an appeal to (2.11) and (2.12) it is evident that

$$e^{cC} e^{kB} f(x, y) = (1 - cby)^{-m} \exp \left( -\frac{a cxy}{1 - cby} \right)$$

$$f \left( \frac{xy}{(1 - cby)(y - g + cby)}, \frac{y - g + cby}{1 - cby} \right) \ldots \quad (2.13)$$

where $g$ and $c$ are arbitrary constants and $f(x, y)$ is an arbitrary function.

3. **Part I: Generating Function Derived from the Operator ($A - \nu$)**

Since $y^{\nu} L_{a,b,m,n}(x)$ is a solution of the system

$$Lu = 0 \text{ and } (A - \nu) u = 0.$$ 

We determine generating function of $L_{a,b,m,n}(x)$ by finding

$$e^{kB+CC} [y^{\nu} L_{a,b,m,n}(x)].$$

We need to consider three cases:

**Case I**—Suppose $g = 1$, $c = 0$. In this case we have

$$e^{B} [y^{\nu} L_{a,b,m,n}(x)] = (y - 1)^{\nu} L_{a,b,m,n} \left( -\frac{xy}{y - 1} \right).$$

On the other hand

$$e^{B} [y^{\nu} L_{a,b,m,n}(x)] = \sum_{p=0}^{\infty} \frac{bp}{p!} (1 - m - \nu)_{p} L_{a,b,m,v-p}(x) y^{\nu-p}. $$
Equating the two expressions and replacing \( y^{-1} \) by \( t \), we get

\[
(1 - t)^{\nu} L_{a,b,m,v} \left( \frac{x}{1 - t} \right) = \sum_{p=0}^{\infty} (1 - m - v)_p \ L_{a,b,m,v-p} (x) \frac{(bt)^p}{p!}.
\]

\( \cdots (3.1) \)

**Case II**—Suppose \( g = 0 \), \( c = 1 \). In this case we have

\[
e^c \ [L_{a,b,m,v} (x) \ y^v] = (1 - b \ y)^{-m} \exp \left( \frac{-a \ x \ y}{1 - b \ y} \right)
\]

\[
\times L_{a,b,m,v} \left( \frac{x}{1 - b \ y} \right) \left( \frac{y}{1 - b \ y} \right)^v,
\]

On the other hand

\[
e^c \ [L_{a,b,m,v} (x) \ y^v] = \sum_{p=0}^{\infty} \frac{(v + 1)_p}{p!} \ L_{a,b,m,v+p} (x) \ y^{v+p}
\]

From these two expressions we have

\[
\sum_{p=0}^{\infty} \frac{(v + 1)_p}{p!} \ L_{a,b,m,v+p} (x) \ y^{v+p}
\]

\[
= (1 - b \ y)^{-m-v} \exp \left( \frac{-a \ x \ y}{1 - b \ y} \right) \ L_{a,b,m,v} \left( \frac{x}{1 - b \ y} \right) \ y^v.
\]

If we divide both members of this equation by \( y^v \) and simplify, we get

\[
\sum_{p=0}^{\infty} \frac{(v + 1)_p}{p!} \ L_{a,b,m,v+p} (x) \ y^p
\]

\[
= (1 - b \ y)^{-m-v} \exp \left( \frac{-a \ x \ y}{1 - b \ y} \right) \ L_{a,b,m,v} \left( \frac{x}{1 - b \ y} \right).
\]

\( \cdots (3.2) \)

**Case III**—Suppose \( g \ c \neq 0 \). Without any loss of generality we can choose \( c = 1 \) and \( g = \ -\frac{1}{\omega} \).

\[
e^c \ e^{(-1/\omega)} [y^v \ L_{a,b,m,v} (x)]
\]

*(equation continued on p. 517)*
\[= \omega^{-\nu}(1 - b y)^{-m-\nu}(1 - b y + \omega y)^{\nu} \exp \left( \frac{-a x y}{1 - b y} \right) \]
\[\times L_{a,b,m,v} \left( \frac{x y \omega}{(1 - b y)(1 - b y + \omega y)} \right). \quad \ldots(3.3)\]

On the other hand
\[e^C e^{(-1/m)B} \left[ y^v L_{a,b,m,v}(x) \right] \]
\[= \sum_{q=0}^{\infty} \frac{(C)^q}{q!} \sum_{p=0}^{\infty} \left( -\frac{1}{\omega} \right)^p \frac{B^p}{p!} \left[ L_{a,b,m,v}(x) y^p \right] \]
\[= \sum_{q=0}^{\infty} \frac{(C)^q}{q!} \sum_{p=0}^{\infty} \left( -\frac{1}{\omega} \right)^p \frac{(b)^p}{p!} \left( 1 - m - \nu \right)_p L_{a,b,m,v-p}(x) y^{v-p} \]
\[= \sum_{q=0}^{\infty} \frac{(C)^q}{q!} \sum_{p=0}^{\infty} \left( -\frac{1}{\omega} \right)^p \frac{b^p}{p!} \frac{(v - p + 1)}{q!} \left( 1 - m - \nu \right)_p \]
\[\times L_{a,b,m,v-p+q}(x) y^{v-p+q}. \]

Thus equating the two expressions, we obtain
\[= \omega^{-\nu}(1 - b y)^{-m-\nu}(1 - b y + \omega y)^{\nu} \exp \left( \frac{-a x y}{1 - b y} \right) \]
\[\times L_{a,b,m,v} \left( \frac{x y \omega}{(1 - b y)(1 - b y + \omega y)} \right) \]
\[= \sum_{q=0}^{\infty} \frac{(v - p + 1)}{p!} \frac{b^p}{q!} \left( 1 - m - \nu \right)_p (\omega y)^{v-p} \]
\[\times L_{a,b,m,v-p+q}(x). \quad \ldots(3.4)\]

Again (3.3) may be written as
\[e^C e^{(-1/m)B} \left[ y^v L_{a,b,m,v}(x) \right] \]
\[= (1 - b y)^{-m-\nu} \omega^{-\nu} \exp \left( \frac{-a x y}{1 - b y} \right) (1-b y+\omega y)^{v} \frac{b^v}{v!} \]
\[\times 1F_1 \left[ \begin{array}{c} -\nu; \\ m; \\ a x y \omega \\ b (1 - b y)(1 - b y + \omega y) \end{array} \right]. \]
Then after some simplification, we obtain another generating relation

\[(1 - b y)^{-m \nu} (1 - b y + \omega y)^{\nu} \exp \left( -\frac{a x y}{1 - b y} \right) \times \ _1F_1 \left[ -\nu; m; \frac{a x y \omega}{b (1 - b y)(1 - b y + \omega y)} \right] \]

\[= \sum_{n=0}^{\infty} \ _2F_1 \left[ -n, -\nu; \omega b; \frac{m}{\omega b} \right] L_{\alpha, \beta, \gamma} (x) y^n. \quad \ldots(3.5)\]

4. **Part II: Generating Functions Derived from Operators Not Conjugate to** \((A - \nu)\)

Let \(S = e^{cB} e^{\nu B}\), where \(g\) and \(c\) are arbitrary constants. Now according to Mcbride\(^2\) we find that

\[e^{\nu B} C^{-z^B} = 2 b g A - b g^2 B + C - b g m\]

\[e^B A e^{-\nu B} = A + g B\]

\[e^{cB} A e^{-cB} = A - c C.\]

Consider the set of linear differential operators

\[\{R/R = r_1 A + r_2 B + r_3 C + r_4,\]

for all combinations of zero and nonzero coefficients except for

\[r_1 = r_2 = r_3 = 0.\]

We find that

\[S (A - \nu) S^{-1} = (1 + 2 b c g) A + g B - c (1 + b c g) C + m b c g - \nu.\]

Then for \(r_1 = 1 + 2 b c g, \ r_2 = g, \ r_3 = -c (1 + b c g).\)

We have \(r_1^2 + 4 b r_2 r_3 = 1.\)

Therefore, \(A - \nu\) is not conjugate to operators for which

\[r_1^2 + 4 b r_2 r_3 = 0.\]

We consider the following cases:
Case 1—If \( r_1 = 0, r_2 = 1, r_3 = 0 \), we seek a solution of the system \( Lu = 0 \) and \( (B + \eta) u = 0 \), \( \eta \) is a nonzero constant. For convenience we choose \( \eta = 1 \), and write the equation as

\[
Lu = 0 \text{ and } (B + 1) u = 0.
\]

A solution of this system is

\[
u (x, y) = e^{by} \, _0F_1 \left[ -; m; - axy \right]
\]

If we expand this function, we get

\[
e^{by} \, _0F_1 \left[ -; m; - axy \right] = \sum_{n=0}^{\infty} \frac{L_{a,b,m,n} (x) \, y^n}{(m)_n}.
\]  

... (4.1)

Case 2—r_1 = 2 b, r_2 = 1 and r_3 = - b. We are led to this choice by considering \( e^{\omega C} (B - \omega) e^{-\omega C} \), where \( \omega \) is a nonzero constant. We find that

\[
e^{\omega C} (B - \omega) e^{-\omega C} = 2 \, b \, c \, A + B - b \, c^2 C + (b \, c \, m - \omega).
\]

If we let \( c = 1 \) we get \( r_1 = 2b, \, r_2 = 1 \) and \( r_3 = - b \). Since this choice satisfies the above conditions, we may determine a solution of the system

\[
Lu = 0 \text{ and } (2 b \, A + B - b \, C + b \, m - \omega) u = 0
\]

from the generating function (4.1) by replacing \( y \) by \( - \omega y \)

\[
u (x, - \omega y) = e^{-\omega by} \, _0F_1 \left[ -; m; a \, \omega \, x \, y \right].
\]

We know that for an arbitrary function \( f (x, y) \)

\[
e^{\omega} f (x, y) = b^m \left( 1 - b \, y \right)^{-m} \exp \left( \frac{- a \, x \, y}{1 - b \, y} \right) \frac{x}{1 - b \, y} \frac{y}{1 - b \, y} \right).
\]

Then

\[
e^{\omega} u (x, - \omega y) = (1 - by)^{-m} \exp \left( \frac{- a \, x \, y}{1 - b \, y} \right) \exp \left( \frac{- b \, \omega \, y}{1 - b \, y} \right)
\]

\[
\times \, _0F_1 \left[ -; m; \frac{a \, \omega \, x \, y}{(1 - by)^2} \right]
\]

\[
= (1 - b \, y)^{-m \omega} \left[ \exp \left( \frac{- a \, x \, y}{1 - b \, y} \right) \sum_{k=0}^{\infty} \frac{L_{a,b,m,k}}{(m)_k} \right]
\]

\[
\times \, \frac{x}{1 - b \, y} \frac{y}{1 - b \, y} \right)^{-\omega y} \right).
\]
using (4.1).

With the help of (3.2) we get

\[
\sum_{n=0}^{\infty} n! \, L_{a,b,m,n}(x) \, L_{a,b,m,n}(b \omega/a) \, \frac{(y/b)^n}{(m)_n} \]

\[
= (1 - b \, y)^{-m} \, \exp \left( - \frac{(a \, x + \omega \, b) \, y}{1 - b \, y} \right) \, _0F_1 \left[ -; m; \frac{a \omega \, x \, y}{(1 - b \, y)^2} \right].
\]

...(4.2)

**Case 3** - \( r_1 = 0, \ r = 0, \ r_3 = 1 \). We seek the solution of the system

\[
Lu = 0 \quad \text{and} \quad (C + \lambda) \, u = 0
\]

\( \lambda \) is a nonzero constant.

We may avoid actually solving this system by noting that

\[
2b \, c \, (1 + b \, c \, g) \, A + (1 + b \, c \, g)^2 \, B
\]

\[
- b \, c^2 \, C + m \, c \, b \, (1 + b \, c \, g) + 1.
\]

If we choose

\( g = 1/b \) and \( c = -1 \),

we get

\[
\frac{1}{e^b} \, e^{-C} \, (B + 1) \, e^{c \, e^{-b}} = -b \, C + 1.
\]

Therefore, we can obtain a solution of \( Lu = 0 \) and \((b \, C - 1) \, u = 0\) by transforming generating function (4.1) as

\[
\frac{1}{e^b} \, e^{-C} \left[ e^{b \, y} \, _0F_1 \left[ -; m; -axy \right] \right]
\]

\[
= y^{-m} \, \exp \left( - \frac{a \, x}{b} \right) \, \exp \left( \frac{b \, y - 1}{b \, y} \right) \, _0F_1 \left[ -; m; - \frac{a \, x}{b^2 \, y} \right].
\]
If we let \( y = -1/t \) and expand in power of \( t \) we get
\[
e^{\frac{t}{tb}} \binom{a}{0} F_1 \left[ \begin{array}{c} n + m; \\ a x/b \\ \frac{b^2}{m} \end{array} \right] = \sum_{n=0}^{\infty} e^{-a x/b} F_1 \left[ \begin{array}{c} n + m; \\ a x/b \\ m; \end{array} \right] \frac{(t/b)^n}{n!},
\]

By Kummer's first formula Rainville\textsuperscript{3}, [p. 125, Theorem 42]
we have
\[
e^{-a x/b} F_1 \left[ \begin{array}{c} n + m; \\ a x/b \\ m; \end{array} \right] = F_1 \left[ \begin{array}{c} -n; \\ a x/b \\ m; \end{array} \right].
\]

Therefore we obtained the generating relation
\[
e^{\frac{t}{tb}} \binom{a}{0} F_1 \left[ \begin{array}{c} -; \\ a x t/b^2 \\ m; \end{array} \right] = \sum_{n=0}^{\infty} 1 F_1 \left[ \begin{array}{c} -n; \\ -a x/b \\ m; \end{array} \right] \frac{(t/b)^n}{(m)_n}.
\]
\[
= \sum_{n=0}^{\infty} L_{a,b,m,n} (-x) \frac{(t/b)^n}{(m)_n} \quad \ldots(4.3)
\]

But with \( x \) replaced by \(-x\) and \( t \) by \( b^2 t \) we have exactly the same generating relation already in Case 1.

**REFERENCES**