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# GROUP-THEORETIC ORIGINS OF CERTAIN GENERATING FUNCTIONS OF THE MODIFIED LAGUERRE POLYNOMIALS

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In this paper L. Weisner's group theoretic method of obtaining generating functions is utilized in the case of modified Laguerre polynomials $L_{a,b,m,n}(x)$, by giving suitable interpretation to the index $n$, in order to derive the elements of Lie-algebra.

## 1. INTRODUCTION

Recently Goyal[1] has defined the modified Laguerre polynomials of degree $n$ as follows :

$$L_{a,b,m,n}(x) = \frac{b^n (m)_n}{n!} {}_1F_1(-n; m; a x/b).$$

We note the following results for this polynomials set

$$DL_{a,bm,n}(x) = \frac{1}{bx}\{b(1 - m - n) L_{a,b,m,n-1}(x) + nL_{a,b,m,n}(x)\} \quad ...(1.1)$$

$$DL_{a,b,m,n}(x) = -\frac{1}{bx}[\{b(-m-n) + a x\} L_{a,b,m,n}(x) + (n+1) L_{a,b,m+n+1}(x)] \quad ...(1.2)$$

where

$$D \equiv \frac{d}{dx}$$

and

$$XD^2 L_{a,b,m,n}(x) + (m - ax/b) DL_{a,b,m,n}(x) + \frac{a}{b} nL_{a,b,m,n}(x) = 0. \quad ...(1.3)$$

The object of the present paper is to derive some generating functions, which are believed to be new of the modified Laguerre polynomials, by suitable interpreating $n$ with the help of Weisner's group theoretic method.

## 2. Group-theoretic Discussion

In order to use Weisner's method we construct from (1.3) following partial differential equation by replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$ ,$n$ by $y\frac{\partial}{\partial y}$ and $L_{a,b,m,n}(x)$ by $u$, $(x,y)$ :

$$x\ \frac{\partial^2 u}{\partial x^2} + (m - a\,x/b)\ \frac{\partial u}{\partial x} + \frac{a}{b}\ y\ \frac{\partial u}{\partial y} = 0. \qquad \ldots(2.1)$$

Let $L$ represent the differential operator of (2.1) i. e.,

$$L = x\ \frac{\partial^2}{\partial x^2} + (m - a\,x/b)\ \frac{\partial}{\partial x} + \frac{a}{b}\ y\ \frac{\partial}{\partial y}\,. \qquad \ldots(2.2)$$

We now find two linear differential operators $B$ and $C$. Let

$$B = B_1(x, y)\ \frac{\partial}{\partial x} + B_2(x, y)\ \frac{\partial}{\partial y} + B_0(x, y)$$

with the aid of (1.1), we have

$$\begin{aligned} B\,[L_{a,b,m,n}(x)\ y^n] = {} & B_1 y^n x^{-1}\,[b\,(1-m-n)\ L_{a,b,m,n-1}(x) \\ & + nL_{a,b,m,n}(x)] + B_2\,[ny^{n-1}]\ L_{a,b,m,n}(x) \\ & + B_0 L_{a,b,m,n}(x)\ y^n. \end{aligned} \qquad \ldots(2.3)$$

In order to make the coefficient of $L_{a,b,m,n-1}(x)\ y^{n-1}$ independent of $x$ and $y$ we choose $B_1 = xy^{-1}$. Then (2.3) becomes

$$\begin{aligned} B\,[L_{a,b,m,n}(x)\ y^n] = {} & b\,(1 - m - n)\ L_{a,b,m,n-1}(x)\ y^{n-1} + L_{a,b,m,n}(x)\ y^n \\ & \times [y^{-1}\,(n + B_2\,n) + B_0]. \end{aligned} \qquad \ldots(2.4)$$

In (2.4) we can make the coefficient of $L_{a,b,m,n}(x)\ y^n$ equal to zero by choosing $B_2 = -1$ and $B_0 = 0$.

Therefore

$$B = x\ y^{-1}\ \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \qquad \ldots(2.5)$$

and

$$B\,[L_{a,b,m,n}(x)\ y^n] = b\,(1 - m - n)\ L_{a,b,m,n-1}(x)\ y^{n-1}. \qquad \ldots(2.6)$$

Similarly it is fairly easy to observe that

$$C = bxy \frac{\partial}{\partial x} + by^2 \frac{\partial}{\partial y} + (b m - a x) y \qquad ...(2.7)$$

and consequently

$$C [L_{a,b,m,n}(x) y^n] = (n + 1) L_{a,b,m,n+1}(x) y^{n+1}. \qquad ...(2.8)$$

To find the group of operators, let us write $A = y (\partial/\partial y)$.

Then we have the following commutator relations,

$$[A, B] = - B, [A, C] = C, [B, C] = - 2 bA - bm. \qquad ...(2.9)$$

We would like to prove that these operators commute with $L$ or $\psi(x) L$, where $\psi(x)$ be a function of $x$ yet to be determined. We express $L$ or $\psi(x) L$ in terms of these operators we know that

$$Lu = x \frac{\partial^2 u}{\partial x^2} + (m - a x/b) \frac{\partial u}{\partial x} + \frac{a}{b} y \frac{\partial u}{\partial y} .$$

Also in making the computation of $[B, C]$, we find that

$$CBu = bx^2 \frac{\partial^2 u}{\partial x^2} - b y^2 \frac{\partial^2 u}{\partial y^2} + (a x - bm) \left( - x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right).$$

Again if we take

$\psi(x) = bx$, we have

$$[bx L - CB] u = b y^2 \frac{\partial^2 u}{\partial y^2} + b m y \frac{\partial u}{\partial y}$$

but

$$b A^2 u = b y \frac{\partial}{\partial y} \left( y \frac{\partial u}{\partial y} \right).$$

Therefore

$$[b xL - C B] u = (b A^2 + (m - 1) bA) u$$

or equivalenty,

$$bxL u = (CB + b A^2 + (m - 1) b A) u. \qquad ...(2.10)$$

It follows from the relations (2. 9) and (2.10) and may be verified by direct calculation, that $bxL$ is commutative with $A$, $B$, $C$ and hence with $R = r_1 A + r_2 B + r_3 C + r_4$ where $r$'s are arbitrary constants, $R$ the set of linear differential operators.

The commutator relation (2.9) shows that the operator 1, $A$, $B$, $C$ generates a Lie algebra. The element of this Lie algebra may be represented in the form $e^R$.

This Lie algebra determines a root system and a Weyl group. The extended form of the group generated by $B$ and $C$ is given by

$$e^{cC} f(x, y) = (1 - cby)^{-m} \exp\left(\frac{-acxy}{1 - cby}\right) f\left(\frac{x}{1 - cby}, \frac{y}{1 - cby}\right) \quad \ldots(2.11)$$

$$e^{gB} f(x, y) = f\left(\frac{xy}{y - g}, y - g\right). \quad \ldots(2.12)$$

By an appeal to (2.11) and (2.12) it is evident that

$$e^{cC}\, e^{gB} f(x, y) = (1 - cby)^{-m} \exp\left(\frac{-acxy}{1 - cby}\right) f\left(\frac{xy}{(1 - cby)(y - g + cbgy)}, \frac{y - g + cbgy}{1 - cby}\right) \quad \ldots(2.13)$$

where $g$ and $c$ are arbitrary constants and $f(x, y)$ is an arbitrary function.

## 3. Part I : Generating Function Derived from the Operator $(A - \nu)$

Since $y^\nu L_{a,b,m,\nu}(x)$ is a solution of the system

$$Lu = 0 \text{ and } (A - \nu)\, u = 0.$$

We determine generating function of $L_{a,b,m,n}(x)$ by finding

$$e^{gB+cC}\,[y^\nu L_{a,b,m,\nu}(x)].$$

We need to consider three cases :

*Case I*—Suppose $g = 1$, $c = 0$. In this case we have

$$e^B\,[y^\nu L_{a,b,m,\nu}(x)] = (y - 1)^\nu L_{a,b,m,\nu}\left(\frac{xy}{y - 1}\right).$$

On the other hand

$$e^B\,[y^\nu L_{a,b,m,\nu}(x)] = \sum_{p=0}^{\infty} \frac{b^p}{p!} (1 - m - \nu)_p\, L_{a,bm,\nu-p}(x)\, y^{\nu-p}.$$

Equating the two expressions and replacing $y^{-1}$ by $t$, we get

$$(1-t)^{\nu} L_{a,b,m,\nu}\left(\frac{x}{1-t}\right) = \sum_{p=0}^{\infty} (1-m-\nu)_p \, L_{a,b,m,\nu-p}(x) \, \frac{(bt)^p}{P!} \, . \qquad \ldots(3.1)$$

*Case II*—Suppose $g = 0$, $c = 1$. In this case we have

$$e^{C}\left[L_{a,b,m,\nu}(x)\, y^{\nu}\right] = (1-by)^{-m} \exp\left(\frac{-axy}{1-by}\right) \times L_{a,b,m,\nu}\left(\frac{x}{1-by}\right)\left(\frac{y}{1-by}\right)^{\nu},$$

On the other hand

$$e^{C}\left[L_{a,b,m,\nu}(x)\, y^{\nu}\right] = \sum_{p=0}^{\infty} \frac{(\nu+1)_p}{p!} \, L_{a,b,m,\nu+p}(x)\, y^{\nu+p}.$$

From these two expressions we have

$$\sum_{p=0}^{\infty} \frac{(\nu+1)_p}{p!} \, L_{a,b,m,\nu+p}(x)\, y^{\nu+p} = (1-by)^{-m-\nu} \exp\left(\frac{-axy}{1-by}\right) L_{a,b,m,\nu}\left(\frac{x}{1-by}\right) y^{\nu}.$$

If we divide both members of this equation by $y^{\nu}$ and simplify, we get

$$\sum_{p=0}^{\infty} \frac{(\nu+1)_p}{p!} \, L_{a,b,m,\nu+p}(x)\, y^{p} = (1-by)^{-m-\nu} \exp\left(\frac{-axy}{1-by}\right) L_{a,b,m,\nu}\left(\frac{x}{1-by}\right). \qquad \ldots(3.2)$$

*Case III*—Suppose $g\,c \neq 0$. Without any loss of generality we can choose $c = 1$ and $g = -\frac{1}{\omega}$.

$$e^{C}\, e^{(-1/\omega)}\left[y^{\nu} L_{a,b,m,\nu}(x)\right]$$

*(equation continued on p. 517)*

$$= \omega^{-\nu}(1 - b\,y)^{-m-\nu}(1 - b\,y + \omega\,y)^{\nu} \exp\left(\frac{-a\,x\,y}{1 - b\,y}\right)$$

$$\times L_{a,b,m,\nu}\left(\frac{x\,y\,\omega}{(1 - b\,y)(1 - b\,y + \omega\,y)}\right). \qquad \ldots(3.3)$$

On the other hand

$$e^{C}\, e^{(-1/\omega)B}\,[y^{\nu} L_{a,b,m,\nu}(x)]$$

$$= \sum_{q=0}^{\infty} \frac{(C)^q}{q!} \sum_{p=0}^{\infty} \frac{\left(-\frac{1}{\omega}\right)^p}{p!} B^p\,[L_{a,b,m,\nu}(x)\, y^{\nu}]$$

$$= \sum_{q=0}^{\infty} \frac{(C)^q}{q!} \sum_{p=0}^{\infty} \frac{\left(-\frac{1}{\omega}\right)^p}{p!} (b)^p\,(1 - m - \nu)_p\, L_{a,b,m,\nu-p}(x)\, y^{\nu-p}$$

$$= \sum_{q=0}^{\infty} \sum_{p=0}^{\infty} \frac{\left(-\frac{1}{\omega}\right)^p b^q}{p!} \frac{(\nu - p + 1)}{q!} (1 - m - \nu)_p$$

$$\times L_{a,b,m,\nu-p+q}(x)\, y^{\nu-p+q}.$$

Thus equating the two expressions, we obtain

$$(1 - b\,y)^{-m-\nu}(1 - b\,y + \omega y)^{\nu} \exp\left(\frac{-a\,x\,y}{1 - b\,y}\right)$$

$$L_{a,b,m,\nu}\left(\frac{x\,y\,\omega}{(1 - b\,y)(1 - by + \omega y)}\right)$$

$$= \sum_{q=0}^{\infty} \frac{(\nu - p + 1)_c}{p!}\, y^q \sum_{p=0}^{\infty} \frac{(-b)^p\,(1 - m - \nu)_p}{q!} (\omega\,y)^{\nu-p}$$

$$\times L_{a,b,m,\nu-p+q}(x). \qquad \ldots(3.4)$$

Again (3.3) may be written as

$$e^{C}\, e^{(-1/\omega)B}\,[y^{\nu} L_{a,b,m,\nu}(x)]$$

$$= (1 - b\,y)^{-m-\nu}\,\omega^{-\nu} \exp\left(\frac{-a\,x\,y}{1 - b\,y}\right)(1 - b\,y + \omega\,y)^{\nu} \frac{b^{\nu}\,(m)_{\nu}}{\nu!}$$

$$\times {}_1F_1\left[\begin{matrix} -\nu; \\ m; \end{matrix} \quad \frac{a\,x\,y\,\omega}{b\,(1 - b\,y)(1 - b\,y + \omega\,y)}\right].$$

Then after some simplification, we obtain another generating relation

$$(1 - b\,y)^{-m-\nu}\,(1 - b\,y + \omega\,y)^{\nu} \exp\left(\frac{-a\,x\,y}{1 - b\,y)}\right)$$

$$\times\ {}_1F_1\left[-\nu;\ m;\ \frac{a\,x\,y\,\omega}{b\,(1 - b\,y)\,(1 - b\,y + \omega\,y)}\right]$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, -\nu; \omega/b \\ m; \end{matrix}\right] L_{a,b,m,n}\,(x)\,y^n. \qquad \ldots(3.5)$$

## 4. Part II : Generating Functions Derived from Operators not Conjugate to $(A - \nu)$

Let $S = e^{cC}\,e^{gB}$, where $g$ and $c$ are arbitrary constants. Now according to Mcbride[2] we find that

$$e^{gB}\,C^{-gB} = 2\,bgA\ - b\,g^2\,B + C - bgm$$

$$e^{gB}\,A\,e^{-gB} = A + gB$$

$$e^{cC}\,A\,e^{-cC} = A - cC.$$

Consider the set of linear differential operators

$$\{R/R = r_1\,A + r_2\,B + r_3\,C + r_4,$$

for all combinations of zero and nonzero coefficients except for

$$r_1 = r_2 = r_3 = 0\}.$$

We find that

$$S\,(A-\nu)\,S^{-1} = (1 + 2\,bcg)\,A + g\,B - c\,(1 + bcg)\,C + m\,bcg - \nu.$$

Then for $r_1 = 1 + 2\,b\,c\,g,\ r_2 = g,\ r_3 = -\,c\,(1 + b\,c\,g)$.

We have $r_1^2\ + 4\,b\,r_2\,r_3 = 1$.

Therefore, $A - \nu$ is not conjugate to operators for which

$r_1^2\ + 4\,b\,r_2\,r_3 = 0.$

We consider the following cases :

*Case* 1—If $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, we seek a solution of the system $Lu = 0$ and $(B + \eta)\, u = 0$, $\eta$ is a nonzero constant. For convenience we choose $\eta = 1$, and write the equation as

$$Lu = 0 \text{ and } (B + 1)\, u = 0.$$

A solution of this system is

$$u(x, y) = e^{by}\, {}_0F_1\,[-;\, m;\, -axy]$$

If we expand this function, we get

$$e^{by}\, {}_0F_1\,[-;\, m;\, -a\,x\,y] = \sum_{n=0}^{\infty} \frac{L_{a,b,m,n}(x)\, y^n}{(m)_n} \qquad \ldots(4.1)$$

*Case* 2—$r_1 = 2\,b$, $r_2 = 1$ and $r_3 = -\,b$. We are led to this choice by considering $e^{cC}\,(B - \omega)\, e^{-cC}$, where $\omega$ is a nonzero constant. We find that

$$e^{cC}\,(B - \omega)\, e^{-cC} = 2\,b\,c\,A + B - b\,c^2C + (b\,c\,m - \omega).$$

If we let $c = 1$ we get $r_1 = 2b$, $r_2 = 1$ and $r_3 = -\,b$. Since this choice satisfies the above conditions, we may determine a solution of the system

$$Lu = 0 \text{ and } (2\,b\,A + B - b\,C + b\,m - \omega)\, u = 0$$

from the generating function (4.1) by replacing $y$ by $-\,\omega y$

$$u(x, -\,\omega\, y) = e^{-\omega by}\, {}_0F_1\,[-;\, m;\, a\,\omega xy].$$

We know that for an arbitrary function $f(x, y)$

$$e^{C} f(x, y) = b^m\,(1 - b\,y)^{-m} \exp\left(\frac{-\,a\,x\,y}{1 - b\,y}\right) f\left(\frac{x}{1 - b\,y}, \frac{y}{1 - b\,y}\right).$$

Then

$$e^{C}\, u(x, -\,\omega y) = (1 - by)^{-m} \exp\left(\frac{-a\,x\,y}{1 - b\,y}\right) \exp\left(\frac{-\,b\,\omega\, y}{1 - b\,y}\right)$$

$$\times\, {}_0F_1\left[-;\, m;\ \frac{a\,\omega\,x\,y}{(1 - by)^2}\right]$$

$$= (1 - b\,y)^{-m-k} \exp\left(\frac{-\,a\,x\,y}{1 - b\,y}\right) \sum_{k=0}^{\infty} \frac{L_{a,b,m,k}\left(\dfrac{x}{1 - b\,y}\right)(-\omega\ y)^k}{(m)_k}$$

using (4.1).

With the help of (3.2) we get

$$\sum_{n=0}^{\infty} n!\, L_{a,b,m,n}(x)\, L_{a,b,m,n}(b\,\omega/a)\, \frac{(y/b)^n}{(m)_n}$$

$$= (1 - b\,y)^{-m} \exp\left( \frac{-(a\,x + \omega\,b)\,y}{1 - b\,y} \right) {}_0F_1\left[ -;\, m;\, \frac{a\,\omega\,x\,y}{(1 - b\,y)^2} \right]. \qquad \ldots(4.2)$$

*Case* 3— $r_1 = 0, \quad r = 0, r_3 = 1$. We seek the solution of the system

$$Lu = 0 \text{ and } (C + \lambda)\,u = 0$$

$\lambda$ is a nonzero constant.

We may avoid actually solving this system by noting that

$$e^{gB}\, e^{cC}\, (B + 1)\, e^{-cC}\, e^{-gB}$$

$$= 2\,b\,c\,(1 + b\,c\,g)\,A + (1 + b\,c\,g)^2\,B$$

$$- b\,c^2\,C + m\,c\,b\,(1 + b\,c\,g) + 1.$$

If we choose

$$g = 1/b \text{ and } c = -1,$$

we get

$$e^{\frac{1}{b}B}\, e^{-C}\, (B + 1)\, e^{C}\, e^{-\frac{1}{b}B} = -b\,C + 1.$$

Therfore, we can obtain a solution of $L\,u = 0$ and $(b\,C - 1)\,u = 0$ by transforming generating function (4.1) as

$$e^{\frac{1}{b}B}\, e^{-C}\, [e^{by}\, {}_0F_1\,[\, -;\, m;\, -axy]]$$

$$= y^{-m} \exp\left( \frac{a\,x}{b} \right) \exp\left( \frac{b\,y - 1}{b\,y} \right) {}_0F_1\left[ \begin{matrix} - ; \\ m; \end{matrix} \; - \frac{a\,x}{b^2\,y} \right].$$

If we let $y = -1/t$ and expand in power of $t$ we get

$$e^{t/b}\,{}_0F_1\left[\begin{matrix}-;\\ m;\end{matrix}\quad \frac{a\,x\,t}{b^2}\right]$$

$$= \sum_{n=0}^{\infty} e^{-ax/b}\,{}_1F_1\left[\begin{matrix}n+m;\\ m;\end{matrix}\quad a\,x/b\right]\frac{(t/b)^n}{n!},$$

By Kummer's first formula Rainville[3], [p. 125, Theorem 42]

we have

$$e^{-ax/b}\,{}_1F_1\left[\begin{matrix}n+m;\\ m;\end{matrix}\quad a\,x/b\right] = {}_1F_1\left[\begin{matrix}-n;\\ m;\end{matrix}\quad -a\,x/b\right].$$

Therefore we obtained the generating relation

$$e^{t/b}\,{}_0F_1\left[\begin{matrix}-;\\ m;\end{matrix}\quad a\,x\,t/b^2\right] = \sum_{n=0}^{\infty} {}_1F_1\left[\begin{matrix}-n;\\ m;\end{matrix}\quad -a\,x/b\right]\frac{(t/b)^n}{(m)_n}.$$

$$= \sum_{n=0}^{\infty} \frac{L_{a,b,m,n}(-x)\,(t/b^2)^n}{(m)_n}. \qquad \ldots(4.3)$$

But with $x$ replaced by $-x$ and $t$ by $b^2\,t$ we have exactly the same generating relation already in Case 1.

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