HYDROMAGNETIC FLOW OF AN INCOMPRESSIBLE VISCOUS
CONDUCTING FLUID BETWEEN TWO PERMEABLE BEDS

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At present we know relatively little about the flow of fluids between two
permeable beds. Because of its intrinsic importance in many industrial pro-
blems and its relevance to the general natural phenomena, in this paper, we
have chosen to study the flow of fluid between two permeable beds. A
physical model illustrating the problem under consideration consists of a
parallel channel of width $h$ bounded by two permeable beds. The lower
bed moves with a uniform velocity $U$ while the upper one is at rest. To
discuss the solution, the flow region is divided into three Zones. The flow
region in the upper bed is called Zone 1 and is governed by Darcy’s law. The
flow region between the two beds is named as Zone 2 and the flow is governed
by Navier-Stokes equation. The flow region in the lower bed is termed as Zone
3 and is governed by Darcy’s law. The expressions for velocity and tempera-
ture distributions are obtained and the effects of magnetic parameter, slip
parameter and porous parameters are investigated on velocity and tempera-
ture distributions. We have obtained the expressions for the thicknesses of
the boundary layer in Zones 1 and 3.

1. Introduction

Flows through porous media are very much prevalent in nature and hence their
study is of principal interest in many scientific and engineering applications. This type
of flows are of great importance in chemical engineering (for filtration and water puri-
fication processes) and petroleum engineering (for studying the movement of natural
gas, oil and water through the oil reservoirs). To study the under ground water
resources, and seepage of water in river beds, one needs to investigate the flows of fluids
through porous media. In the case of flow past a porous medium Beavers and Joseph have shown that the usual no slip condition at the porous boundaries is no longer valid and they have postulated that slip exists, called $BJ$ slip condition, at the porous boundaries because of the transfer of momentum. Later Taylor and Saffman studied this $BJ$ slip condition. The Poiseuille flow over a permeable bed has been discussed by Beavers and Joseph, Rajasekhara et al. have investigated Couette flow over a per-
meable bed with an impermeable moving plate, using the slip boundary conditions at
the lower permeable bed. Venugopal and Bathaiah\textsuperscript{a} have studied the flow of viscous incompressible and slightly conducting fluid past a permeable bed in three different configurations namely (1) Couette flow; (2) Poiseuille flow; and (3) Free surface flow under the influence of a uniform transverse magnetic field. Srinivasan and Bathaiah\textsuperscript{a} have studied the hydromagnetic channel flow of conducting incompressible fluid of variable viscosity between two parallel walls with non-erodable porous lining. Bathaiah and Venugopal\textsuperscript{b} have studied the effect of porous lining on the MHD flow between two concentric rotating cylinders under the influence of a uniform transverse magnetic field. As present we know relatively little about the flow of fluids between two permeable beds. Because of its intrinsic importance in many industrial problems and its relevance to the general natural phenomena, in this paper we have chosen to study the flow of fluid between two permeable beds. We have evaluated the velocity and temperature distributions. The effects of magnetic, slip and porous parameters on velocity and temperature distributions are investigated.

2. FORMULATION AND SOLUTION OF THE PROBLEM

A physical model (Figure A) illustrating the problem under consideration consists of a parallel channel of width ‘\(h\)’ bounded by two permeable beds. The lower bed moves with a uniform velocity \(U\) while the upper one is at rest. To discuss the solution, the flow region is divided into three Zones. The flow region in the upper bed is called Zone 1 and is governed by Darcy’s law. The flow region between the two beds is named as

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{A}
\caption{A physical model.}
\end{figure}
Zone 2 and the flow is governed by Navier-Stokes equations. The flow region in the lower bed is termed as Zone 3. The lower nominal surface is taken as $X$-axis and a straight line perpendicular to that as the $Y$-axis. We introduce the magnetic field of intensity $H_0$ in the $Y$-direction. The fluid is viscous incompressible and slightly conducting. Therefore the magnetic Reynolds number is much less than unity and hence the induced magnetic field can be neglected in comparison with the applied magnetic field. To derive the basic equations for the present model, we make the following assumptions:

1. The flow in the $x$-direction is driven by a constant pressure gradient $\frac{\partial p}{\partial x}$.

2. The fluid is Newtonian viscous incompressible and slightly conducting.

3. The flow is steady and fully developed with negligible body forces so that all the physical quantities except the pressure are functions of ‘$y’ only.

4. The lower and upper permeable beds are homogeneous and non-conducting with constant permeabilities $K_1$ and $K_2$ respectively.

Under these assumptions and in the absence of any input electric field the basic equations of the flow in the three Zones are given below:

\[
\frac{d^2 u}{dy^2} - \frac{u}{K_1} - \sigma \frac{H_0^2}{\mu} u = \frac{1}{\mu} \frac{\partial p}{\partial x} \quad \quad \text{...(2.1)}
\]

\[
K_T \frac{d^2 T}{dy^2} + \mu \left( \frac{d u}{dy} \right)^2 + \frac{\mu}{K_1} u^2 + \sigma \frac{H_0^2}{\mu} u^2 = 0 \quad \quad \text{...(2.2)}
\]

where $K_1 = K_2$ in Zone 1, $K_1 = \alpha$ in Zone 2, and $K_1 = K_3$ in Zone 3, $u$ is the fluid velocity in the $X$-direction, $\sigma$ the electrical conductivity of the fluid, $\mu_r$ the magnetic permeability, $\mu$ the coefficient of viscosity, $p$ the fluid pressure, $T$ the temperature and $K_T$ the thermal conductivity of the fluid.

We introduce the following non-dimensional quantities:

\[
u^* = \frac{u}{U_0} ; \quad y^* = \frac{y}{h} ; \quad \chi^* = \frac{x}{h} ; \quad p^* = \frac{p}{\rho U_0^2} ; \quad U^* = \frac{U}{U_0} ;
\]

\[
T^* = \frac{T - T_{B1}}{T_{B2} - T_{B1}} ; \quad T_{B1}^* = \frac{T_{B1} - T_1}{T_{B2} - T_{B1}} ; \quad T_{B2}^* = \frac{T_{B2} - T_1}{T_{B2} - T_{B1}} \quad \text{...(2.3)}
\]

where $T_{B1}$, $T_{B2}$ are the slip temperatures at $y = 0$ and $y = h$ respectively and $T_1$ the ambient temperature in Zone 1 and Zone 3.
In view of equation (2.3) the equations (2.1) and (2.2) reduce to (after dropping superscripts "*").

\[
\frac{d^2u}{dy^2} - C^2_i u = -P \tag{2.4}
\]

\[
\frac{d^2T}{dy^2} = -P, \quad E \left[ \left( \frac{du}{dy} \right)^2 + C^2_i u^2 \right] \tag{2.5}
\]

where \( C^2_i = C^2 = a^2_2 + M \) in Zone 1, \( C^2_i = M \) in Zone 2,

\[
C^2_i = C^2_1 = a^2_1 + M \text{ in Zone 3.}
\]

\[
M = \frac{\sigma \nu_s^2 H_0^2}{\mu} \quad \text{(Magnetic parameter)}
\]

\[
a_2 = \frac{h}{K_2^{1/2}}; \quad P = -R \frac{dp}{dx}
\]

\[
P_r = \frac{\mu C_p}{K_T} \quad \text{(Prandtl number)}
\]

\[
E = \frac{U_0^2}{C_p (T_{B2} - T_{B1})} \quad \text{(Eckert number)}
\]

\[
a_1 = \frac{h}{K_1^{1/2}}.
\]

The non-dimensional boundary conditions are

**Zone 1**

\[
u = u_{B2}, \quad T = 1 \text{ at } y = 1 \tag{2.6a}
\]

\[
u = \frac{P}{a^2_2}, \quad T = -T_{B1} \text{ at } y = 1 + n_2 \tag{2.6b}
\]

**Zone 2**

\[
u = u_{B1} + U, \quad T = 0 \text{ at } y = 0 \tag{2.6c}
\]
\[ u = u_{B2}, \ T = 1 \text{ at } y = 0 \] 

\[ \frac{du}{dy} = s \ a_1 \left( u_{B1} - \frac{p}{a_1^2} \right), \quad \frac{dT}{dy} = \beta \ a_1 \ T_{B1} \text{ at } y = 0 \] 

\[ \frac{du}{dy} = -s \ a_1 \left( u_{B2} - \frac{p}{a_2^2} \right), \quad \frac{dT}{dy} = -\beta \ a_2 \ T_{B2} \text{ at } y = 1. \] 

\[ \text{Zone 3} \]

\[ u = u_{B1} + U, \ T = 0 \text{ at } y = 0 \] 

\[ u = \frac{p}{a_1^2}, \ T = -T_{B1} \text{ at } y = -n_1 \]

where \( u_{B1}, \ u_{B2} \) are the slip velocities at \( y = 0 \) and \( y = h \) respectively, \( s \) is the slip parameter, \( \beta \) the Biot number and \( n_1 \) and \( n_2 \) are the boundary layer thicknesses in Zones 3 and 1 respectively.

Solving the equations (2.4) and (2.5) using the boundary conditions (2.6), we obtain velocity and temperature distributions in all the three Zones.

**Zone 1**

\[ u_1 = \frac{1}{\sinh Cn_2} \left[ d_1 \sinh C (1 - y + n_2) + d_2 \sinh C (1 - y) \right] + \frac{p}{C^2} \]

\[ T_1 = 1 + \frac{(1 + T_{B1}) (1 - y)}{n_2} - \frac{P_r \ E C^2}{4 C^2 \sinh^2 Cn_2} \left[ d_1^2 \cosh 2C (1 - y + n_2) + d_2^2 \cosh 2C (1 - y) + 2 d_1 d_2 \cosh C (2 - 2y + n_2) + 4 C^2 d_2 d_3 \sinh C (1 - y + n_2) \right. \]

\[ + \frac{y}{n_2} \left( (\cosh 2 Cn_2 - 1) \left( d_1^2 - d_2^2 \right) + 4 C^2 d_3 d_4 \sinh Cn_2 \right) \]

\[ - \frac{1}{n_2} \left( d_1^2 (1 + n_2) (\cosh 2 Cn_2 - 1 + d_2^2 ((1 + n_2) - \cosh 2 Cn_2) + 2 d_1 d_2 n_2 \cosh Cn_2 + 4 C^2 \sinh Cn_2 (n_2 d_1 + d_4)) \right]. \]

\[ \text{...(2.8)} \]
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Zone 2

\[
\begin{align*}
  u_2 &= \frac{1}{\sinh M^{1/2}} \left[ d_8 \sinh M^{1/2} y + d_6 \sinh M^{1/2} (1 - y) \right] + \frac{P}{M} \
  T_1 &= y - \frac{P_r E}{4 M \sinh^2 M^{1/2}} \left[ d_6^2 M \cosh 2 M^{1/2} y + d_6^2 M \cosh 2 M^{1/2} \
  &\quad \times (1 - y) - 2 d_5 d_6 \cosh M^{1/2} (1 - 2 y) + 2 P^2 y^2 \sinh^2 M^{1/2} \
  &\quad + 4 P d_6 \sinh M^{1/2} \sinh M^{1/2} (1 - y) + 4 P d_6 \sinh M^{1/2} \sinh M^{1/2} y \
  &\quad - 4 M^2 C_b y - 4 M^2 C_4 \right] \
  \end{align*}
\]

when the lower boundary is also at rest, i.e. \( U = 0 \), the velocity and temperature distributions are

\[
\begin{align*}
  u_0 &= -\frac{1}{\sinh M^{1/2}} \left[ d_8 \sinh M^{1/2} y + d_6 \sinh M^{1/2} (1 - y) \right] + \frac{P}{M} \
  T_0 &= y - \frac{P_r E}{4 M \sinh^2 M^{1/2}} \left[ - d_6^2 M \cosh 2 M^{1/2} y \
  &\quad + d_7^2 M \cosh 2 M^{1/2} (1 - y) - 2 d_7 d_6 M \cosh M^{1/2} (1 - 2 y) \
  &\quad + 2 P^2 y^2 \sinh^2 M^{1/2} + 4 P d_7 \sinh M^{1/2} \sinh M^{1/2} (1 - y) \
  &\quad + 4 P d_6 \sinh M^{1/2} \sinh M^{1/2} y - 4 M^2 C_b^* y - 4 M^2 C_4^* \right] \
  \end{align*}
\]

where \( u_{B_0}, \ u_{B_0}^* \) are now the lower and upper slip velocities respectively

\[
\begin{align*}
  u_{B_0} &= \frac{1}{C_2} \left[ \left( \frac{P}{a_2^2} - \frac{P}{M} \right) \frac{s a_2 M^{1/2}}{\sinh M^{1/2}} + \frac{P}{M} \left( M \coth^2 M^{1/2} \
  &\quad s a_2 M^{1/2} \coth M^{1/2} - \frac{M}{\sinh^2 M^{1/2}} \right) + (s a_2 + M^{1/3} \coth M^{1/3}) \frac{5P}{a_1} \right] \
  u_{B_0}^* &= \frac{1}{C_2} \left[ \left( \frac{P}{a_1^2} - \frac{P}{M} \right) \frac{s a_1 M^{1/2}}{\sinh M^{1/2}} + \frac{P}{M} \left( s a_1 M^{1/2} \coth M^{1/2} \right) \right] \
  \end{align*}
\]

(equation continued on p. 528)
\[ + \ M \ coth^2 M^{1/2} - \frac{M}{\sinh^2 M^{1/2}} \] \[ + (sa_1 + M^{1/2} \coth M^{1/2}) \]
\[ \frac{sp}{a_2} \]
\[ \left( \frac{2}{4M} \left[ \frac{d_8^2 - d_7^2}{\cosh 2M^{1/2}} \right] + \frac{P \sinh^2 M^{1/2}}{2M^2} \left( P + 4d_8 - 4d_7 \right) \right) \]
\[ C_3^* = \frac{1}{4M^2} \left[ d_8^2 \ M + d_7^2 \ M \cosh 2M^{1/2} - 2d_7^2 \ M \cosh M^{1/2} \right. \]
\[ + 8P \sinh^2 M^{1/2} \] \[ + \left( \frac{2}{4M} \left[ \frac{d_8^2 - d_7^2}{\cosh 2M^{1/2}} \right] + \frac{P \sinh^2 M^{1/2}}{2M^2} \left( P + 4d_8 - 4d_7 \right) \right) \]
\[ C_4^* = \frac{1}{4M^2} \left[ d_8^2 \ M + d_7^2 \ M \cosh 2M^{1/2} - 2d_7^2 \ M \cosh M^{1/2} \right. \]
\[ + 8P \sinh^2 M^{1/2} \]

**Zone 3**

\[ u_3 = \frac{1}{\sinh C_1 n_1} \left[ d_9 \ \sinh C_1 (n_1 + y) + d_{10} \ \sinh C_1 y \right] \] \[ + \ 2d_9 \ d_{10} \ \cosh C_1 (n_1 + 2y) + 4 \ d_9 \ d_{11} \ C_1^2 \ \sinh C_1 (n_1 + y) \]
\[ + 4d_{10} \ d_{11} C_1^2 \ \sinh C_1 y + \frac{y}{n_1} \left\{ \left( \cosh 2C_1 n_1 - 1 \right) \right. \]
\[ \times \left( \frac{d_{10}^2 - d_9^2}{\cosh 2C_1 - d_{10} + d_9} \right) - 4d_{11} C_1^2 \ \sinh C_1 n_1 (d_{10} + d_9) \}
\[ - \left\{ d_9^2 \ \cosh 2C_1 n_1 + d_{10}^2 + 2d_9 d_{10} \ \cosh C_1 n_1 \right. \]
\[ + 4d_9 d_{11} C_1^2 \ \sinh C_1 n_1 \}
\[ \left\} \right] \] \[ \ldots (2.16) \]

Where

\[ u_{B1} = \frac{1}{C_2} \left[ \left( \frac{P}{a_2^2} - \frac{P}{M} \right) \frac{sa_2}{\sinh M^{1/2}} + \left( \frac{P}{M} - U \right) (M \ coth^2 M^{1/2} \]
\[ + sa_2 \ M^{1/2} \coth M^{1/2} - \frac{M}{\sinh^2 M^{1/2}} \right) \]
\[ + \left( sa_2 + M^{1/2} \coth M^{1/2} \right) \frac{sp}{a_1} \] \[ \left. \right] \] \[ \ldots (2.17) \]
\[ u_{B2} = \frac{1}{C_2} \left[ \left( \frac{P}{a_2^2} - \frac{P}{M} + U \right) \frac{sa_1}{\sinh M^{1/2}} \frac{M^{1/2}}{M^{1/2}} + \frac{P}{M} \left( sa_1 \cdot M^{1/2} \coth M^{1/2} \right. \right. \]
\[ \left. \left. + M \coth^2 M^{1/2} - \frac{M}{\sinh^2 M^{1/2}} \right) + (sa_1 + M^{1/2} \coth M^{1/2}) \frac{5P}{a_2} \right] \]
\[ \cdots (2.18) \]

\[ T_{B1} = -\frac{1}{\beta a_1} + \frac{P_r EM}{2 \beta a_1 M^{3/2} \sinh^2 M^{1/2}} \left[ d_6^2 M \sinh 2M^{1/2} \right. \]
\[ - 2Md_5 d_6 \sinh M^{1/2} + 2d_6 \sinh 2M^{1/2} - 4Pd_6 \sinh M^{1/2} \]
\[ + 2C_3 M^{3/2} \]

\[ T_{B2} = -\frac{1}{\beta a_2} + \frac{P_r EM}{2 \beta a_2 M^{3/2} \sinh^2 M^{1/2}} \left[ d_5^2 M^{3/2} \sinh 2M^{1/2} \right. \]
\[ - 2M^{3/2} d_5 d_6 \sinh M^{1/2} + 2P^2 \sinh^3 M^{1/2} - 4d_6 P M^{1/2} \sinh M^{1/2} \]
\[ + 2d_6 M^{1/2} \sinh 2M^{1/2} - 2C_3 M^2 \]

\[ C_2 + (sa_1 + M^{1/2} \coth M^{1/2}) (sa_2 + M^{1/2} \coth M^{1/2}) \frac{M}{\sinh^2 M^{1/2}} \]

\[ C_3 = \frac{(\cosh 2M^{1/2} - 1)}{4M} \left( \frac{d_5^2 - d_6^2}{2} \right) + \frac{P \sinh^2 M^{1/2}}{2M^2} \]

\[ [P + 4(d_6 + d_4)] \]

\[ C_4 = \frac{1}{4M^2} \left[ M \left( d_5^2 + d_6^2 \cdot \cosh 2M^{1/2} - 2d_5d_6 \cosh M^{1/2} \right) \right. \]
\[ \left. + 8 P d_6 \sinh^2 M^{1/2} \right] \]

\[ d_1 = \left( u_{B2} - \frac{P}{C^2} \right) ; \quad d_2 = \left( \frac{P}{C^2} - \frac{P}{a_2^2} \right) ; \quad d_3 = \frac{2P \sinh Cn_2}{C^4} ; \]

\[ d_4 = \left( u_{B2} - \frac{P}{a_2^2} \right) \]

\[ d_5 = \left( u_{B2} - \frac{P}{M} \right) ; \quad d_6 = \left( u_{B1} + U - \frac{P}{M} \right) ; \quad d_7 = \left( u_{B0} - \frac{P}{M} \right) ; \]
\[ d_8 = \left( u_{B0}^* - \frac{P}{M} \right); \quad d_9 = \left( u_{B1} + U - \frac{P}{C_1^2} \right); \]

\[ d_{10} = \left( \frac{P}{C_1^2} - \frac{P}{a_1^2} \right); \quad d_{11} = \frac{2s_P \sinh C_1 n_1}{C_1^4}. \]

**Expressions for the Thickness of the Boundary Layers**

We know that at the edge of the boundary layer, the shear-stress has to be zero. In other words

\[ \frac{du}{dy} = 0 \text{ at } y = n_c. \]

Therefore the expression for \( n_2 \) is given by

\[ n_2 = \left| \frac{C \pm (C^2 - L_2)^{1/2}}{C} \right|. \]  \( \ldots(2.19) \)

Neglecting the terms of \( O\left(n_2^4\right)\)

where

\[ L_3 = (2 + C^2) - \frac{2a_2^2 \left( u_{B2} - P \right) \cosh C}{\left( Pa_2^2 - P C^2 \right)}. \]

Similarly the thickness of the boundary layer \( n_1 \) in Zone 3 is evaluated.

\[ n_1 = \frac{2^{1/2}}{C_1} (L_2 - 1)^{1/2}. \]  \( \ldots(2.20) \)

Neglecting the terms of \( O\left(n_1^4\right)\)

where

\[ L_2 = \frac{a_1^2 \left( (u_{B1} + U) C^2 - P \right)}{\left( P C_1^2 - P a_1^2 \right)}. \]
3. **Particular Cases**

**Case (i)**—Let the lower boundary will be impermeable i.e., \( K_l = K_s = 0 \) \((a_1 \to \infty)\). Then the velocity and temperature distributions are

\[
\begin{align*}
  u^* &= \frac{1}{\sinh M^{1/2}} \left[ d_{12} \sinh M^{1/2} y + d_{13} \sinh M^{1/2} (1 - y) \right] + \frac{P}{M} \\
  T^* &= y - \frac{P_r E}{4M \sinh^2 M^{1/2}} \left[ M \left( d_{12}^2 \cosh 2M^{1/2} y + d_{13}^2 \cosh 2M^{1/2} \times (1 - y) - 2d_{12} d_{13} \cosh M^{1/2} (1 - 2y) \right) + 2P^2 y^2 \sinh^2 M^{1/2} \\
  &+ 8 P d_{13} \sinh M^{1/2} \sinh M^{1/2} (1 - y) + 8 P d_{12} \sinh M^{1/2} \sinh \\
  &\times M^{1/2} y - 4M^2 \left( C_3^{**} y + C_4^{**} \right) \right] \\
\end{align*}
\]

where

\[
\begin{align*}
  u_{b_2}^* &= \frac{1}{C_2^*} \left[ U M + P \left( \frac{s M^{1/2} \sinh M^{1/2}}{a_2} + \cosh M^{1/2} - 1 \right) \right] \\
  C_2^* &= M^{1/2} \sinh M^{1/2} (sa_2 + M^{1/2} \coth M^{1/2}) \\
  C_3^{**} &= \frac{1}{4M^2} \left[ M \left( \cosh 2M^{1/2} - 1 \right) \left( d_{12}^2 - d_{13}^2 \right) + 2P^2 \sinh^2 M^{1/2} \\
  &+ 8 P d_{12} \sinh^2 M^{1/2} - 8 P d_{13} \sinh^2 M^{1/2} \right] \\
  C_4^{**} &= \frac{1}{4M^2} \left[ M \left( d_{12}^2 + d_{13}^2 \cosh 2M^{1/2} - 2d_{12} d_{13} \cosh^2 M^{1/2} \right) \\
  &+ 8 P d_{13} \sinh^2 M^{1/2} \right], \quad d_{12} = u_{b_2}^* - \frac{P}{M}, \quad d_{13} = U - \frac{P}{M} . \\
\end{align*}
\]

**Case (ii)**—Let the upper boundary will be impermeable i.e., \( K_l = K_s = 0 \) \((a_2 \to \alpha)\). Then the velocity and temperature distributions are

\[
\begin{align*}
  u^* &= \frac{1}{M \sinh M^{1/2}} \left[ M d_{14} \sinh M^{1/2} (1 - y) - P \sinh M^{1/2} \right] + \frac{P}{M} \\
  T^* &= y - \frac{P_r E}{4M^2 \sinh^2 M^{1/2}} \left[ \left( P^2 \cosh 2M^{1/2} y + M^2 d_{14}^2 \cosh \right) \right] \text{ (equation continued on p. 532)}
\end{align*}
\]
\[ \times 2M^{1/2} (1 - y) + 2PM d_{14} \cosh M^{1/2} (1 - 2y) \]
\[ + 2P^2 M y^2 \sinh^2 M^{1/2} + 8PMd_{14} \sinh M^{1/2} \sinh M^{1/2} (1 - y) \]
\[ - 8P^2 \sinh M^{1/2} \sinh M^{1/2} y - 4M^2 (C_5 y + C_6) \] ...(3.5)

where
\[ u_{b1}^* = \frac{1}{C_2^*} \left[ P \left( \frac{s M^{1/2} \sinh M^{1/2}}{a_1} + \cosh M^{1/2} - 1 \right) - UM \cosh M^{1/2} \right] \]
...(3.6)
\[ C_5 = \left( \cosh \frac{2M^{1/2} - 1}{4M} \left( \frac{P^2}{M^2} - d_{14}^2 \right) + \frac{1}{2M^2} \left( P^2 M \sinh^2 M^{1/2} \right. \right. \]
\[ - 8P^2 \sinh^2 M^{1/2} - 8Pd_{14} \sinh^2 M^{1/2} \left) \right. \]
\[ C_5 = \frac{1}{4M^2} \left[ P^2 + d_{14}^2 M^2 \cosh 2M^{1/2} + 2PMd_{14} \left( \cosh M^{1/2} \right. \right. \]
\[ + \sinh^2 M^{1/2} \left) \right. \], \[ d_{14} = \left( \frac{u_{b1}^* + U - \frac{P}{M}}{a_1} \right) \]

**Case (iii)—** Let the lower and upper boundary walls be impermeable i.e., \( K_i = K_1 = 0, K_i = K_2 = 0, (a_1 \to \alpha, a_2 \to \alpha) \). Then the velocity and temperature distributions are
\[ u^* = \frac{1}{M \sinh M^{1/2}} \left[ UM \sinh M^{1/2} (1 - y) - P(\sinh M^{1/2} y \right. \]
\[ + \sinh M^{1/2} (1 - y)) \right] + \frac{P}{M} \]
...(3.7)
\[ T^* = \frac{y - \frac{P \epsilon E}{4M^2 \sinh M^{1/2}} \left[ P^2 \cosh 2M^{1/2} + d_{13} M^2 \cosh 2M^{1/2} (1 - y) \right. \]
\[ + 2MPd_{13} \cosh M^{1/2} (1 - 2y) + 2P^2 M y^2 \sinh^2 M^{1/2} \]
\[ + 8PMD_{13} \sinh M^{1/2} \sinh M^{1/2} (1 - y) - 8P^2 \sinh M^{1/2} \sinh M^{1/2} y \]
\[ - 4M^3 \left( \frac{C_5^* y + C_6^*}{a_1} \right) \right] \]
...(3.8)
\[ C_5^* = \frac{(\cosh 2M^{1/2} - 1) \left( \frac{P^2}{M^2} - d_{13}^2 \right) + \frac{P^2 \sinh^2 M^{1/2} (M - 4)}{2M^3} \left. \right. \]
\[ - \frac{2Pd_{13} \sinh^2 M^{1/2}}{M^2} \]
\[ C_\star = \frac{1}{4M^2} [P^2 + d_{15}^2 M^2 \cosh 2M^{1/2} + 2P d_{15} M \cosh M^{1/2} + 4 d_{15} P M \sinh M^{1/2}]. \]

**Case (iv)**—If the permeabilities of both lower and upper permeable walls are equal i.e., \( K_1 = K = K_2 = K \) (i.e., \( a_1 = a_2 = a \)) then the velocity and temperature distribution are

\[ u^* = \frac{1}{\sinh M^{1/2}} [d_{15} \sinh M^{1/2} (1 - y)] + \frac{P}{M} \quad \ldots(3.9) \]

\[ T^* = y - \frac{P \tau E}{4M^2 \sinh^2 M^{1/2}} \left[ M \left( \frac{d_{15}^2 \sinh M^{1/2} y + d_{16}^2 \cosh M^{1/2}}{2M^{1/2} (1 - y) - 2 d_{15} d_{16} \cosh M^{1/2} (1 - 2y)} + 2P^2 y \sinh M^{1/2} \right) \times M^{1/2} + 4P d_{15} \sinh M^{1/2} \cosh M^{1/2} (1 - y) + 4 P d_{15} \sinh M^{1/2} \right] \]

\( \ldots(3.10) \)

where

\[ u^{**}_{B_1} = \frac{1}{C^*_2} \left[ \frac{s a M^{1/2}}{\sinh M^{1/2}} \left( \frac{P}{a^2} - \frac{P}{M} \right) + \left( \frac{P}{M} - U \right) (M \coth^2 M^{1/2} \right. \]

\[ + s a M^{1/2} \coth M^{1/2} \sinh M^{1/2} \] \( \ldots(3.11) \)

\[ u^{**}_{B_2} = \frac{1}{C^*_2} \left[ \frac{s a M^{1/2}}{\sinh^2 M^{1/2}} \left( \frac{P}{a^2} - \frac{P}{M} + U \right) + \frac{P}{M} (s a M^{1/2} \coth M^{1/2} \right. \]

\[ + M \coth^2 M^{1/2} \sinh M^{1/2} \] \( \ldots(3.12) \)

\[ C^*_2 = (s a + M^{1/2} \coth M^{1/2})^2 - \frac{M}{\sinh^2 M^{1/2}} \]

\[ C^*_5 = \frac{1}{4M^2} [M (\cosh 2M^{1/2} - 1) \left( \frac{d_{15}^2}{2} - \frac{d_{16}^2}{2} \right) + 2 P^2 \sinh M^{1/2}] \]

(equation continued on p. 534)
\[ + 4 P d_{18} \sinh^2 M^{1/2} - 4 P d_{16} \sinh^2 M^{1/4} \]

\[ C_{00}^{**} = \frac{1}{4M^2} \left[ \frac{d^2_{18} + d^2_{16} \cosh 2M^{1/2} - 2d_{18} d_{16} \cosh M^{1/2}}{d_{16}} \left( \frac{P}{M} \right) \right] \]

\[ + 8Pd_{16} \sinh^2 M^{1/2} \]

\[ d_{16} = u_{B2}^{**} - \frac{P}{M}, \quad d_{16} = u_{B1}^{**} + U - \frac{P}{M}. \]

Further, if the lower bed is at rest i.e., \( U = 0 \), then the velocity and temperature distributions are

\[ u^* = \frac{d_{17}}{\sinh M^{1/2}} \left[ \sinh M^{1/2} y + \sinh M^{1/2} (1 - y) \right] + \frac{P}{M} \quad \text{...(3.13)} \]

\[ T^* = y - \frac{P \cdot E}{4M \sinh^2 M^{1/2}} \left[ M d_{17}^2 \cosh 2M^{1/2} y + \cosh 2M^{1/2} (1 - y) \right. \]

\[ - 2 \cosh M^{1/2} (1 - 2y) + 8P d_{17} \sinh M^{1/2} (\sinh M^{1/2} (1 - y) \]

\[ \left. + \sinh M^{1/2} y) + 4P^2 \sinh^2 M^{1/2} (y^2 - y) - 4M^2 C_2 C_7 \right] \quad \text{...(3.14)} \]

where

\[ u_{B2}^{**} = u_{B2}^{**} = u_B \]

\[ u_B = \frac{1}{C_{20}^{**}} \left[ \frac{sa M^{1/2}}{\sinh M^{1/2}} \left( \frac{P}{a^2} - \frac{P}{M} \right) + \frac{P}{M} \left( M \coth^2 M^{1/2} \right) \right. \]

\[ + \left. sa M^{1/2} \coth M^{1/2} - \frac{M}{\sinh^2 M^{1/2}} \right) \]

\[ + \left. \frac{sP (sa + M^{1/2} \coth M^{1/2})}{a} \right] \]

\[ C_7 = \frac{d_{17}^2}{4M} \left[ 1 + \cosh 2M^{1/2} - 2 \cosh M^{1/2} \right] + \frac{2P d_{17} \sinh^2 M^{1/2}}{M^2} \]

\[ d_{17} = u_B - \frac{P}{M}. \]
4. Conclusions

Slip Velocities

Figure 1 gives the slip velocity profiles $u_{B1}$. An increase in magnetic parameter $M$ or porosity parameter '$a_1$' causes a decrease in slip velocity $u_{B1}$. In Fig. 2 $u_{B1}$ is drawn against 's' for different values of '$a_2$'. An increase in the slip parameter 's' results in an increase in $u_{B1}$ whereas an increase in the values of porosity parameter $a_2$ results in a decrease in $u_{B1}$. Figure 3 shows the effects of $a_1$ and $M$ on slip velocity $u_{B2}$.

Fig. 1. Slip velocity $u_{B1}$ against $M$ for different values of $a_1$.

Fig. 2. Slip velocity $u_{B1}$ against $s$ for different values of $a_2$.

Fig. 3. Slip velocity $u_{B2}$ against $M$ for different values of $a_1$.

Fig. 4. Slip velocity $u_{B2}$ against $s$ for different values of $a_2$. 
An increase in \( a_1 \) or \( M \) results in a decrease in \( u_B \). Figure 4 is drawn to bring out the effects of \( a_2 \) and \( s \) on \( u_B \). \( u_B \) decreases as \( a_2 \) or \( s \) increases.

**Fig. 5.** Velocity profiles for different values of \( M \).

**Fig. 6.** Velocity profiles for different values of \( a_1 \).

*Velocity distribution*

In Fig. 5 the velocity \( u_2 \) in Zone 2 is plotted against \( y \) for different values of \( M \). As \( M \) increases the velocity \( u_2 \) decreases. Figure 6 illustrates the effects of \( 'a_1' \) on velocity \( u_2 \) in Zone 2. An increase in \( 'a_1' \) results in a decrease in \( u_2 \). Figure 7 brings out the effect of \( 'a_2' \) on \( u_2 \). We have observed that \( u_2 \) decreases as \( 'a_2' \) increases. In Fig. 8, \( u_2 \) is plotted against \( 'y' \) for different values of \( s \). It is noticed that \( u_2 \) increases as

**Fig. 7.** Velocity profiles for different values of \( a_2 \).

**Fig. 8.** Velocity profiles for different values of \( s \).
's' increases up to a certain distance very near to a lower permeable bed (approximately 1/3rd of the length of the Zone 2) and then this trend gets reversed in the remaining flow field.

**Slip Temperature**

In Fig. 9, $T_{B_1}$ is drawn against $M$ for different values of $a_1$. We have observed that $T_{B_1}$ decreases with the increase in $a_1$ whereas it increases as $M$ increases. Figure 10 illustrates the effects of 's' and $a_2$ on $T_{B_1}$. An increase in 's' or 'a_2' results in an increase in $T_{B_1}$. Figure 11 is drawn to bring out the effects of Biot number $\beta$ and $P_rE$ (Product of Prandtl and Eckert number) on $T_{B_1}$. We have seen that $T_{B_1}$ decreases with the increase in $\beta$ whereas $T_{B_1}$ increases as $P_rE$ increases. In Fig. 12, $T_{B_2}$ is drawn against $M$ for different values of 'a_1'. We conclude that $T_{B_2}$ increases with the increase in $M$ whereas $T_{B_2}$ decreases as 'a_1' increase. Figure 13 is drawn to bring out the effect of 's' and 'a_2' on $T_{B_2}$. It is observed that $T_{B_2}$ increases with the increase in 's' whereas $T_{B_2}$ decreases as 'a_2' increases. In Fig. 14, $T_{B_2}$ is drawn against $\beta$ for different values of $P_rE$. It is seen that $T_{B_2}$ decreases with the increase in $\beta$ whereas $T_{B_2}$ increases as $P_rE$ increases.

![Fig. 9](image1.png)  
**Fig. 9.** Slip temperature $T_{B_1}$ against $M$ for different values of $a_1$.  

![Fig. 10](image2.png)  
**Fig. 10.** Slip temperature $T_{B_1}$ against $s$ for different values of $a_2$.  

![Fig. 11](image3.png)  
**Fig. 11.** Slip temperature $T_{B_1}$ against $\beta$ for different of $P_rE$.  

![Fig. 12](image4.png)
Figure 15 brings out the effects of $M$ on temperature $T_2$ in Zone 2. An increase in $M$ results in an increase in $T_2$. Figure 16 gives us the effect of $a_1$ on $T_2$. An increase in $a_1$ causes a decrease in $T_2$. Figure 17 illustrates the effect of $a_2$ on $T_2$. We have seen that $T_2$ decreases with the increases in $a_2$. Figure 18 is drawn to investigate the effect of $s$ on $T_2$. An increase in $s$ causes an increase in $T_2$. An increase in $s$
causes an increase in $T_2$. In Fig. 19, $T_2$ is drawn against $y$ for different values of $Pr \cdot E$. We have observed that $T_2$ increases with the increase in $Pr \cdot E$. Further, it is observed that the temperature profiles is linear when $Pr \cdot E = 0$. Figure 20 brings out the effect $\beta$ on $T_2$. An increase in $\beta$ causes a decrease in $T_2$. 
Fig. 20. Temperature profiles for different values of $\beta$.

REFERENCES