*Indian J. pure appl. Math.*, **17** (1): 58-60, January 1986

# A NOTE ON MAXIMAL AUTOMORPHISM GROUPS OF COMPACT RIEMANN SURFACES

MARSTON CONDER

*Department of Mathematics & Statistics, University of Auckland*
*Auckland, New Zealand*

(*Received* 26 *March* 1985)

This paper ties together the concepts of mark and strong symmetric genus for those finite groups which are representable as the conformal automorphism group of a compact Riemann surface of genus greater than 1, and makes two announcements : first, that the strong symmetric genus of every alternating group and every symmetric group of finite degree is known, and second, that the author has determined all Hurwitz groups of order less than one million.

The purpose of this note is to tie together a couple of concepts which have appeared in some recent papers, and to announce related results which have been obtained by this author, partly in an effort to achieve uniformity and/or to clear any confusion which may exist.

Patra[7] introduced the term 'mark' for a finite group $G$ representable as the conformal automorphism group of a compact Riemann surface of genus $g$ satisfying $|G| \geqq \frac{132}{5}(g-1) > 0$. Specifically, she stated that if $S$ is a compact Riemann surface with genus $g \geqslant 2$, then the first six possible orders (in decreasing order of magnitude) of the group $A(S)$ of conformal automorphisms of $S$ are

$$l_1 = 84(g-1),\ l_2 = 48(g-1),\ l_3 = 40(g-1),$$

$$l_4 = 36(g-1),\ l_5 = 30(g-1),\ l_6 = \frac{132}{5}(g-1)$$

and, correspondingly, the group $A(S)$ is said to have mark 1, 2, 3, 4, 5 or 6. She also pointed out that a finite group $G$ has mark $n$ (with $1 \leqslant n \leqslant 6$) if and only if $G$ can be generated by three elements $x$, $y$ and $z$ satisfying the relations indicated below :

| Mark | Order of $G$ | Relations |
|---|---|---|
| 1 | $84\,(g-1)$ | $x^2 = x^3 = z^7 = xyz = 1$ |
| 2 | $48\,(g-1)$ | $x^2 = y^3 = z^8 = xyz = 1$ |
| 3 | $40\,(g-1)$ | $x^2 = y^4 = z^5 = xyz = 1$ |
| 4 | $36\,(g-1)$ | $x^2 = y^3 = z^9 = xyz = 1$ |
| 5 | $30\,(g-1)$ | $x^2 = y^3 = z^{10} = xyz = 1$ |
| 6 | $\frac{132}{5}(g-1)$ | $x^2 = y^3 = z^{11} = xyz = 1$ |

Strictly speaking, the elements $x$, $y$ and $z$ must have exactly those orders prescribed by the relations—otherwise $G$ could be a spherical space group, even the trivial group !

The same sort of result was given in Tucker[9], a more general paper dealing with group actions on surfaces. Tucker defines the strong symmetric genus $\sigma^0(G)$ of a finite group $G$ to be the minimum genus of those surfaces on which $G$ acts (faithfully) as a group of orientation-preserving homeomorphisms. It is clear that the concepts 'mark' and 'strong symmetric genus' are closely related. Indeed, from the comments following Corollary 13 in Tucker's paper, we see that if $G$ does not act on the sphere or the torus, then the first six possible values (in decreasing order of magnitude) for $|G|/(\sigma^0(G) - 1)$ are 84, 48, 40, 36, 30 and $\frac{132}{5}$, and each of these values is attained only when $G$ is a quotient of the appropriate triangle group.

The correspondence is obvious.

In particular, when (as Patra suggests) we seek the lowest mark attained by a particular finite group, we will actually be trying to find the strong symmetric genus of that group.

If the group $G$ has mark 1, then $G$ is also known as Hurwitz group, and the corresponding surface on which it acts is important in that it has the maximum possible number of conformal automorphisms. Quite a few such groups are known (see refs. 2, 3, 6, 8 for example). If $G$ has mark 2, then Kalita[5] calls $G$ an '$M_2$—group', and refers to $G$ as being a 'second maximal' automorphism group. Similarly, if $G$ has mark 3 then $G$ is called an '$M_3$—group', or 'third maximal'. Obviously one could continue in this way, defining '$M_4$—groups', and so on. In fact a number of finite permutation groups (of small degree) have been considered[1,5,7] in this light.

We now take the opportunity to announce that the strong symmetric genus of every finite alternating group $A_n$ and every finite symmetric group $\Sigma_n$ is known. The main result of Conder[3] shows that all but finitely many $A_n$ are Hurwitz groups (that is, have strong symmetric genus $\frac{n!}{168} + 1$). Note that this means that all but finitely many $A_n$ have mark 1 (not just 'infinitely many', as noted by Patra). Using similar methods, we have found that all but finitely many $\Sigma_n$ have mark 2. In fact $\Sigma_n$ has mark 2 except when $1 \leqslant n \leqslant 17$ or $n = 22, 23, 26$, or 29. (Hence $\Sigma_n$ has strong symmetric genus $\frac{n!}{48} + 1$, for all but 21 values of $n$.) Also it turns out that of the the 64 alternating groups which do not have mark 1, all but 17 have mark 2. The remaining cases can be dealt with easily, by searching for each group amongst the quotients of the triangle groups

$$\Delta(k, l, m) = \langle x, y, z \mid x^k = y^l = z^m = xyz = 1 \rangle$$

with $\frac{1}{k} + \frac{1}{l} + \frac{1}{m}$ taking values in descending order from $\frac{19}{20}$ to $\frac{59}{70}$. Details of these results appear in Conder[4].

Finally (on a different note) the author of this paper has determined all Hurwitz groups of order less than one million. In fact there are just 32 integers $g$ in the range $1 < g \leqslant 11905$ for which there exist compact Riemann surfaces of genus $g$ with $84(g - 1)$ conformal automorphisms. The surfaces of this sort correspond to exactly 92 proper normal subgroups of the triangle group $\Delta(2, 3, 7)$ with index less than $10^6$, each automorphism group being a quotient of $\Delta(2, 3, 7)$ by one (or more) of these normal subgroups. We hope to publish this result in the near future.

## References


1. B. P. Chetiya, and S. C. Kalita, *Indian J. pure appl. Math.* **14** (1983), 830–37.
2. J. M. Cohen, *Math. Proc. Camb. Phil. Soc.* **86** (1979), 395–400.
3. M. D. E. Conder, *J. London Math. Soc.* (2), **22** (1980), 75–86.
4. M. D. E. Conder, *Bull. Austral. Math. Soc.* 30 (1984), 73–90.
5. S. C. Kalita, *Indian J. pure appl. Math.* **10** (1979), 1157–66.
6. A. M. Macbeath, Generators of linear fractional groups, *Number Theory*, Proceedings of Symposia in Pure Mathematics 12 (American Mathematical Society ,Providence, R. I., 1969), pp. 14–32.
7. K. Patra, *Indian J. pure appl. Math.* **14** (1983), 864–72.
8. Chih-han Sah, *Acta Mathematica*, **123** (1969), 13–42.
9. T. W. Tucker, *J. Combin. Theory Ser. B.* **34** (1983), 82–98.