

ON ALMOST STRONGLY θ -UPPER SEMICONTINUOUS MULTIFUNCTIONS

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The aim of this paper is to introduce and study the notion of almost strongly θ -upper semicontinuous multifunctions and their relationship with θ -closed subsets and H -closed subsets.

1. INTRODUCTION

Noiri and Kang¹ have introduced and investigated the concept of almost strongly θ -continuous functions. The purpose of this paper is to extend this concept and some of the results to multifunctions.

A multifunction F of a set X into a set Y is a correspondence such that $F(x)$ is a nonempty subset of Y for each $x \in X$, that is it is a function $F: X \rightarrow p(Y) - \{\phi\}$, where $p(Y)$ is the power set of Y . We will denote such a multifunction by $F: X \rightarrow Y$. The graph $G(F)$ of F is the subset $\{(x, y) \mid x \in X, y \in F(x)\}$ of $X \times Y$. Let X and Y be topological spaces. For a subset A of X , the θ -closure of A , denoted by $\text{cl}_\theta A$, is the subset $\{x \in X \mid \text{cl}(V) \cap A \neq \phi \text{ for each neighbourhood } V \text{ of } x\}$. If $\text{cl}_\theta A = A$, then A is called θ -closed. The graph $G(F)$ of $F: X \rightarrow Y$ is said to be θ -closed if it is a θ -closed subset of $X \times Y$. It is easy to see that $G(F)$ is θ -closed if and only if for each $(x, y) \in X \times Y - G(F)$ there exist open subsets $U \subset X$ and $V \subset Y$ containing x and y , respectively, such that $F(\text{cl}(U)) \cap \text{cl}(V) = \phi$. A multifunction $F: X \rightarrow Y$ is called θ -upper semicontinuous (θ -u.s.c) (Joseph²) if for each $x \in X$ and each open set $V \subset Y$ containing $F(x)$, there exists an open $U \subset X$ containing x such that $F(\text{cl}(U)) \subset \text{cl}(V)$.

A space X is said to be almost-regular if for each regular open set V and each $x \in V$, there exists an open set U such that $x \in U \subset \text{cl}(U) \subset V$. A subset K of a topological space X is called quasi H -closed relative to X if for each cover $\{G_\alpha \mid \alpha \in \Delta\}$ of K by n open subsets of X , there exists a finite subfamily $\{G_{\alpha_i} \mid i = 1, \dots, n\}$ such that $K \subset \bigcup_{i=1}^n \text{cl}(G_{\alpha_i})$. If X is quasi H -closed relative to X , then it is called quasi H -closed. Quasi H -closed Hausdorff spaces are usually called H -closed.

2. ALMOST STRONGLY θ -UPPER SEMICONTINUOUS
MULTIFUNCTIONS

Definition—A multifunction $F: X \rightarrow Y$ is said to be almost strongly θ -upper semicontinuous (briefly a.s. θ -u.s.c) if for each $x \in X$ and each open set $V \subset Y$ containing $F(x)$, there exists an open set $U \subset X$ containing x such that $F(\text{cl}(U)) \subset \text{int}(\text{cl}(V))$.

Theorem 2.1—For a multifunction $F: X \rightarrow Y$ the following are equivalent :

(a) F is a.s. θ -u.s.c

(b) For each regular closed subset K of Y , $F^{-1}(K) = \{x \in X \mid F(x) \cap K \neq \phi\}$ is θ -closed in X .

(c) For each $x \in X$ and each regular open set $V \subset Y$ containing $F(x)$, there exists an open set $U \subset X$ containing x such that $F(\text{cl}(U)) \subset V$.

PROOF : (a) \Rightarrow (b). Let K be a regular closed subset of Y and $x \in \text{cl}_\theta F^{-1}(K)$. If $x \notin F^{-1}(K)$, then $F(x) \cap K = \phi$. Hence $F(x) \subset Y - K$. Since $Y - K$ is a regular open set and F is a.s. θ -u.s.c. there exists an open set $U \subset X$ containing x such that $F(\text{cl}(U)) \subset \text{int}(\text{cl}(Y - K)) = Y - K$. Thus $F(\text{cl}(U)) \cap K = \phi$, that is, $\text{cl}(U) \cap F^{-1}(K) = \phi$. This contradicts with $x \in \text{cl}_\theta F^{-1}(K)$.

(b) \Rightarrow (c). Let $x \in X$ and V be any regular open subset of Y containing $F(x)$. By (b), $F^{-1}(Y - V)$ is θ -closed in X and $x \notin F^{-1}(Y - V)$. Since $F^{-1}(Y - V)$ is θ -closed in X there exists an open set U containing x such that $\text{cl}(U) \cap F^{-1}(Y - V) = \phi$. This implies that $F(\text{cl}(U)) \subset V$.

(c) \Rightarrow (a). Let $x \in X$ and V be an open subset of Y containing $F(x)$. $\text{int}(\text{cl}(V))$ is a regular open set containing $F(x)$. By (c), there exists an open set U containing x such that $F(\text{cl}(U)) \subset \text{int}(\text{cl}(V))$. Hence F is a.s. θ -u.s.c.

It is clear that every a.s. θ -u.s.c multifunction is θ -u.s.c, but the converse need not be true as the following example shows.

Example—Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{c\}, \{a, b\}, \{a, b, c\}\}$. Let $Y = \{p, q, r, s\}$ and $\sigma = \{\phi, Y, \{p\}, \{q\}, \{p, q\}\}$. Define a multifunction $F: (X, \tau) \rightarrow (Y, \sigma)$ as follows :

$$F(a) = F(b) = p, F(c) = r, F(d) = \{r, s\}.$$

Then F is θ -u.s.c, but it is not a.s. θ -u.s.c.

The following theorem shows that, under certain hypotheses, a θ -u.s.c multifunction is a.s. θ -u.s.c.

Theorem 2.2—Let $F : X \rightarrow Y$ be a multifunction such that $F(x)$ is quasi H -closed relative to Y for each $x \in X$. If F is θ -u.s.c and Y is an almost-regular space then F is a.s. θ -u.s.c.

PROOF : Let $x \in X$ and $F(x) \subset V$, where V is a regular open set in Y . It is easy to show that if K is quasi H -closed relative to an almost-regular space Y and V is a regular open set containing K , then there exists an open set W such $K \subset W \subset \text{cl}(W) \subset V$. Since $F(x)$ is quasi H -closed relative to Y and V is regular open, there is an open set W such that $F(x) \subset W \subset \text{cl}(W) \subset V$. Since F is θ -u.s.c, there exists an open set U containing x such that $F(\text{cl}(U)) \subset \text{cl}(W)$. Thus $F(\text{cl}(U)) \subset V$, that is, F is a.s. θ -u.s.c.

Theorem 2.3—Let $F : X \rightarrow Y$ be a multifunction with compact point images (that is, $F(x)$ is compact for each $x \in X$). Then F is a.s. θ -u.s.c and X is almost-regular if and only if the graph map $G_F : X \rightarrow X \times Y$, defined by $G_F(x) = \{(x, y) \mid y \in F(x)\}$, is a.s. θ -u.s.c.

PROOF : Suppose F is a.s. θ -u.s.c and X is almost-regular. Let $x \in X$ and G be a regular open subset of $X \times Y$ containing $G_F(x)$. It is easy to show that for regular open set G containing compact set $G_F(x)$, there exist regular open sets $V \subset X$ and $W \subset Y$ such that $G_F(x) \subset V \times W \subset G$. Since F is a.s. θ u.s.c there exists an open set $U \subset X$ containing x such that $F(\text{cl}(U)) \subset W$. Since X is almost-regular U can be chosen such that $x \in U \subset \text{cl}(U) \subset V$. Now $G_F(\text{cl}(U)) \subset \text{cl}(U) \times W \subset V \times W \subset G$. Hence G_F is a.s. θ -u.s.c.

Conversly, suppose $G_F : X \rightarrow X \times Y$ is a.s. θ -u.s.c. Let $x \in X$ and V be a regular open subset of Y containing $F(x)$, that is, $F(x) \subset V$. Let $P_2 : X \times Y \rightarrow Y$ be the second projection. Since P_2 is continuous and open $P_2^{-1}(V)$ is a regular open subset of $X \times Y$ and $G_F(x) \subset P_2^{-1}(V)$. Since G_F is a.s. θ -u.s.c, there exists an open set $U \subset X$ containing x such that $G_F(\text{cl}(U)) \subset P_2^{-1}(V)$. Now it is easy to see that $F(\text{cl}(U)) \subset V$. Hence F is a.s. θ -u.s.c.

Now we will show that X is almost-regular space. Let V be a regular open set in X and $x \in V$. $G_F(x) = \{(x, y) \mid y \in F(x)\} \subset V \times Y$ and $V \times Y$ is a regular open set in $X \times Y$. Since G_F is a.s. θ -u.s.c, there exists an open subset U of X containing x such that $G_F(\text{cl}(U)) \subset V \times Y$. Now it is clear that $x \in U \subset \text{cl}(U) \subset V$. Hence X is almost-regular.

Theorem 2.4—If $E, F : X \rightarrow Y$ are a.s. θ -u.s.c multifunctions with compact point images and Y is Hausdorff, then $A = \{x \in X \mid E(x) \cap F(x) \neq \phi\}$ is a θ -closed subset of X .

PROOF : Let $z \notin A$. Then $E(z) \cap F(z) = \phi$. In a Hausdorff space disjoint compact subsets are separated by disjoint open sets. Hence there are open sets V and W such that $E(z) \subset V$, $F(z) \subset W$ and $V \cap W = \phi$. $V \cap W = \phi$ implies $\text{int}(\text{cl}(V)) \cap \text{int}(\text{cl}(W)) = \phi$. On the other hand, since E and F are a.s. θ -u.s.c, there are open sets U_1, U_2 both containing z such that $E(\text{cl}(U_1)) \subset \text{int}(\text{cl}(V))$ and $F(\text{cl}(U_2)) \subset \text{int}(\text{cl}(W))$. Let us put $U = U_1 \cap U_2$. Then U is an open neighbourhood of z and $\text{cl}(U) \cap A = \phi$. Otherwise for any $t \in \text{cl}(U) \cap A$, we have $E(t) \subset \text{int}(\text{cl}(V))$, $F(t) \subset \text{int}(\text{cl}(W))$ and $E(t) \cap F(t) \neq \phi$. This gives $\text{int}(\text{cl}(V)) \cap \text{int}(\text{cl}(W)) \neq \phi$ which is a contradiction. Hence A is a θ -closed subset of X .

Corollary—If $F : X \rightarrow Y$ is an a.s. θ -u.s.c multifunction with compact point images and X is an almost-regular Hausdorff space, then the set $A = \{x \in X \mid x \in F(x)\}$ is a θ -closed subset of X .

PROOF : If X is an almost-regular space then the identity function is an a.s. θ -u.s.c multifunction.

The following theorem is a slight improvement of Theorem 2.3—Corollary in Özer³.

Theorem 2.5—Let $F : X \rightarrow Y$ be a multifunction with compact point images and Y an H -closed space. The following are equivalent :

- (a) F is a.s. θ -u.s.c,
- (b) F has θ -closed graph,
- (c) For each subset K , H -closed relative to Y , $F^{-1}(K)$ is θ -closed in X
- (d) For each H -closed subset K of Y , $F^{-1}(K)$ is θ -closed in X .

PROOF : (a) \Rightarrow (b). Let $(x, y) \in X \times Y - G(F)$. Since $F(x)$ is compact, $y \notin F(x)$ and Y is Hausdorff there are open sets V and W of Y such that $y \in V$, $F(x) \subset W$ and $V \cap W = \phi$. This gives $\text{cl}(V) \cap \text{int}(\text{cl}(W)) = \phi$. By (a), there exists an open set U containing x such that $F(\text{cl}(U)) \subset \text{int}(\text{cl}(W))$. Hence $F(\text{cl}(U)) \cap \text{cl}(V) = \phi$, that is, $G(F)$ is θ -closed subset of $X \times Y$.

(b) \Rightarrow (c). Let K be an H -closed subset relative to Y and $x \in X - F^{-1}(K)$. For each $y \in K$, $(x, y) \notin G(F)$ and hence there are open sets $U_y \subset X$ and $V_y \subset Y$ containing x and y , respectively, such that $\text{cl}(U_y \times V_y) \cap G(F) = \phi$. Now $\{V_y \mid y \in K\}$ is a cover of K by open subsets of Y . Hence for some $y_1, y_2, \dots, y_n \in K$, $K \subset \bigcup_{i=1}^n \text{cl}(V_{y_i})$. Let us put $U = \bigcap_{i=1}^n U_{y_i}$. U is an open set containing x and $F(\text{cl}(U)) \cap K = \phi$. This implies that $\text{cl}(U) \cap F^{-1}(K) = \phi$, consequently $x \notin \text{cl}_\theta(F^{-1}(K))$.

(c) \Rightarrow (d). Since an H -closed subset of Y is H -closed relative to Y (the converse need not be true), this implication is clear.

(d) \Rightarrow (a). Let $x \in X$ and V a regular open subset of Y such that $F(x) \subset V$. $Y - V$ is a regular closed set and since Y is an H -closed space, $Y - V$ is H -closed. By (d), $F^{-1}(Y - V)$ is θ -closed in X and $x \notin F^{-1}(Y - V)$. Hence there exists an open set U containing x such that $\text{cl}(U) \cap F^{-1}(Y - V) = \emptyset$. This implies that $F(\text{cl}(U)) \subset V$, that is F is a.s. θ -u.s.c which completes proof.

Remark: In the proof (a) \Rightarrow (b) in the above theorem we used only the Hausdorff condition on Y . Hence one can say that an a.s. θ -u.s.c multifunction with compact point images into a Hausdorff space has a θ -closed graph. The implication (b) \Rightarrow (c) can be shown without the assumption that $F(x)$ is compact for each $x \in X$ and Y is H -closed.

Theorem 2.6—Let $F: X \rightarrow Y$ be a multifunction and Y a quasi H -closed space. If F has θ -closed graph, then F is a.s. θ -u.s.c.

PROOF: Let K be a regular closed subset of Y . Since Y is quasi H -closed, regular closed set K is quasi H -closed. Using the implication (b) \Rightarrow (c) in Theorem 2.5 (see Remark), we see that $F^{-1}(K)$ is θ -closed and hence by Theorem 2.1, F is a.s. θ -u.s.c.

The following theorem can be proved making small modifications to the proof of Theorem 2.5 in (Özer³).

Theorem 2.7—If $F: X \rightarrow Y$ is an a.s. θ -u.s.c multifunction with compact point images and A is quasi H -closed relative to X , then $F(A)$ is quasi H -closed relative to Y .

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