Indian J. pure appl. Math., 17 (1) : 66-73, January 1986

ON FINSLER SPACES OF CONSTANT CURVATURE

B. B. SINHA AND A. S. MATHAROO
Department of Mathematics, Banaras Hindu University, Varanasi 221005

(*Received* 11 *March* 1985; *after revision* 17 *June* 1985)

In the present paper, we have constructed a tensor $W^{(2)'h}_{ijk}$ and has shown its invariance under a C-projective mapping and a Finsler space is of constant curvature if and only if $W^{(2)'h}_{ijk} = 0$. We hav econsidered the conditions for aFinsler space of scalar curvature to turn into a space of constant curvature.

1. INTRODUCTION

The concept of Finsler space of scalar curvature and of constant curvature was introduced by Berwald[1]. The characteristics form of curvature tensor of such a space was obtained by Shibata[5] who showed that a Finsler space F^n ($n \geq 2$) of non-vanishing scalar curvature with vanishing stretch curvature tensor is a Riemannian space of constant curvature. Recently, Sakaguchi[4] and Fukui and Yamada[2,3] investigated certain conditions under which a Finsler space of scalar curvature turns into the space of constant curvature.

The purpose of the present paper is to investigate certain conditions so that a Finsler space of scalar curvature turns into a Finsler space of constant curvature. For this, a tensor $W^{(2)'h}_{ijk}$ has been defined and the necessary and sufficient conditions that a Finsler space of scalar curvature becomes a space of constant curvature are obtained.

2. PRELIMINARIES

Let F^n be an n-dimensional Finsler space with the fundamental function $L(x, y)$, the positive definite metric $g_{ij} = \frac{1}{2}\dot{\partial}_j \dot{\partial}_i L^2$ and the angular metric tensor $h_{ij} = g_{ij} - l_i l_j$ where $\dot{\partial}_i = \dfrac{\partial}{\partial y^i}$ and $l_i = \dot{\partial}_i L$.

The Cartan and Berwald covariant derivatives for a vector v^i with respect to x^k and y^k are defined by

$$v^i_{|k} = d_k v^i + {}^* \Gamma^i_{jk} v^j, \quad v^i_{|k} = \dot\partial_k v^i + C^i_{jk} v^j$$

$$v^i_{(k)} = d_k v^i + G^i_{jk} v^j, \quad v^i_{\|k} = \dot\partial_k v^i$$

where $d_k = \partial_k - N^m_\kappa \dot\partial_m$, $\partial_k = \dfrac{\partial}{\partial x^k}$, $G^i_k = \dot\partial_k G^i$ and $G^i = \frac{1}{2} y^i_{00}$,

the index 0 stands for transvection by y and

$$\gamma^i_{jk} = \tfrac{1}{2} g^{ih} (\partial_k g_{jh} + \partial_j g_{kh} - \partial_h g_{jk})$$

$${}^* \Gamma^i_{jk} = \tfrac{1}{2} g^{ih} (d_k g_{jh} + d_j g_{kh} - d_h g_{jk})$$

$$G^i_{jk} = \dot\partial_k G^i_j, \quad C_{ijk} = \tfrac{1}{2} \dot\partial_k g_{ij}$$

The curvature and torsion tensors are defined as

$$R^i_{hjk} = \left(d_k {}^* \Gamma^i_{hj} + {}^* \Gamma^m_{hj} {}^* \Gamma^i_{mk} - j \mid k \right) + C^i_{hm} H^m_{jk}$$

$$P_{hijk} = \left(C_{ijk|h} + C^m_{hj} P_{mik} - h \mid i \right) = \dot\partial_h P_{ijk} + P^m_{hj} C_{mik} - h \mid i$$

$$S^i_{hjk} = - C^m_{hj} C^i_{mk} - j \mid k, \quad P^i_{jk} = C^i_{jk|0} = P^i_{0jk}$$

$$H^i_{jk} = d_k G^i_j - j \mid k = R^i_{0jk} = H^i_{0jk}$$

$$H^i_{hjk} = d_k G^i_{hj} + G^m_{hj} G^i_{mk} - j \mid k, \quad H^i_k = H^i_{0k}$$

$$G^i_{hjk} = \dot\partial_k G^i_{hj}, \quad H^i_{jk} = \frac{1}{3} \dot\partial_j H^i_k - j \mid k$$

where $j \mid k$ stands for interchange of j and k.

We denote by $p.$ the projection on the indicatrix. For example, for the tensor T^i_j, $p.\ T^i_j = h^i_a\ h^b_j\ T^a_b$. If a tensor T satisfies $p.T = T$, the tensor T is called an indicatric tensor.

3. A Finsler Space of Scalar Curvature

A Finsler space F^n $(n > 2)$ of scalar curvature K $(K \neq 0)$ is characterized by[4]

$$H_k^i = L^2 K h_k^i \qquad \ldots(3.1)$$

or

$$H_{jk}^i = L\left(K l_j + \frac{1}{3} K_j\right) h_k^i - j \mid k \qquad \ldots(3.2)$$

or

$$H_{hjk}^i = K g_{hj} \delta_k^i + \frac{1}{3}\left\{h_{hj} l^i K_k + h_h^i l_j K_k + h_k^i \left(K_{hj} + l_h K_j + 2K_h l_j\right)\right\} - j \mid k \qquad \ldots(3.3)$$

where $K_j = L \dot{\partial}_j K, K_{jl} = L p. (\dot{\partial}_j K_l) = K_{lj}$.

In fact

$$\begin{aligned} K_{jl} &= L p. (\dot{\partial}_j K_l) \\ &= L K_{(a)} h_l^a h_j^b \\ &= L \left(\dot{\partial}_b K_l + L^{-1} K_a h_b^a l_l\right) h_j^b \\ &= L \left(\dot{\partial}_b K_l + L^{-1} l_l K_b\right)\left(\delta_j^b - l^b l_j\right) \\ &= L \dot{\partial}_j K_l + l_l K_j \\ &= K_{lj}. \end{aligned}$$

From (3.3), we have

(a) $H_{hj} := H_{hjl}^l = (n-1) K g_{hj} + \frac{1}{3}\{(n-2)(K_{hj} + l_h K_j) + (2n-1) K_h l_j\}$.

(b) $H_{ijk}^l = H_{kj} - j \mid k = \frac{1}{3}(n+1) K_k l_j - j \mid k$

(c) $H_{l0} = L\{(n-1) K l_l + \frac{1}{3}(2n-1) K_l\}$

(d) $H_i = H_{0i} = L\{(n-1)K l_i + \frac{1}{3}(n-2)K_i\}$

(e) $H_{00} = (n-1)KL^2$, $K_0 = 0$, $\dot{\partial}_k K_i y^k = 0$, $\dot{\partial}_i K_0 = -K_i$. ...(3.4)

Also Sakaguchi[4] has shown that a Finsler space of scalar curvature turns into a space of constant curvature if and only if

(a) $K_i = 0$, (b) $K_{ij} = 0$. ...(3.5)

A diffeomorphism of a Finsler space to another Finsler space is called an projective mapping if it maps an arbitrary geodesics on a geodesics[3]. We shall be concerned with a diffeomorphism of F^n to another Finsler space $\bar{F}^n = (\bar{M}^n, \bar{L})$. We can also use local coordinates x^i and y^i of F^n for expressing various quantities of $\bar{F}^n$, because $\phi : F^n \to \bar{F}^n$ is diffeomorphic and it is seen that $\phi : F^n \to \bar{F}^n$ satisfies

$$\frac{dx^i}{dt} = y^i, \frac{dy^i}{dt} + \gamma^i_{00} = \alpha y^i \qquad ...(3.6)$$

$$2G^i = \gamma^i_{00}, \; G^i_j = \dot{\partial}_j G^i, \; G^i_{jk} = \dot{\partial}_k G^i_j \qquad ...(3.7)$$

where γ^i_{jk} are Christoffel's symbols constructed from g_{ij}. ϕ is a projective mapping if and only if there exists a positive homogeneous scalar field $P(x, y)$ of degree one satisfying

$$\bar{G}^i = G^i + P y^i \qquad ...(3.8)$$

where P is called the projective factor of the projective mapping. From (3.8) we can obtain[3]

$$\bar{H}^h_{ijk} = H^h_{ijk} + y^h \dot{\partial}_i Q_{jk} + \delta^h_i Q_{jk} + \delta^h_j \dot{\partial}_i Q_k - \delta^h_k \dot{\partial}_i Q_j$$

where $H^h_{ijk} = \dot{\partial}_i R^h_{jk}$ and $R^h_{jk} = \dfrac{\partial G^h_j}{\partial x^k} - G^h_{jm} G^m_k - j \mid k$ and

$$Q_i = P_{(i)} - P p_i, \; p_i = \dot{\partial}_i P \qquad ...(3.9)$$

$$Q_{ij} = P_{i(j)} - P_{j(i)}. \qquad ...(3.10)$$

Also Fukui and Yamada[3] proved that a projective mapping of a Finsler space of constant curvature is also of constant curvature if and only if the projective factor p satisfies the differential equation

$$Q_{ij} = P_{i(j)} - P_{j(i)}. \qquad ...(3.11)$$

Such a projective mapping considered above is called a C-projective mapping. Under C-projective mapping the following equations hold

$$\overset{h}{\bar{H}}_{ijk} = H^h_{ijk} + \delta^h_j \dot{\partial}_i Q_k - \delta^h_i \dot{\partial}_i Q_j \qquad ...(3.12)$$

$$\dot{\partial}_j Q_i - \dot{\partial}_i Q_j = 0 \qquad ...(3.13)$$

where Q_i is a certain vector field positive homogeneous of degree one. From (2.12) and (3.13), we have

$$\bar{H}_{ij} = H_{ij} - (n-1)\,\partial_i Q_j \qquad ...(3.14)$$

where $H_{ij} = H^m_{ijm}$.

4. Certain Conditions for a Finsler Space to be of Constant Curvature

Let us define a tensor

$$W^{(2)'h}_{ijk} = H^h_{ijk} + \frac{1}{(n+1)}\left\{ y^h \dot{\partial}_i H_{jk} + \frac{(n+1)}{(n-1)} \delta^h_j H_{ik} - j \mid k \right\}. \qquad ...(4.1)$$

Firstly, we shall prove the following :

Theorem 4.1—A Finsler space is of constant curvature if and only if

$$W^{(2)'h}_{ijk} = 0.$$

Proof : Assume that $W^{(2)'h}_{ijk} = 0$. Contracting (3.1) by $y^i y'$, we have $R^h_{0k} = \frac{1}{(n-1)}\left\{ y^h H_{0k} - H_{00} \delta^h_k \right\}$. Contracting further with y_h, we have $H_{0k} = L^{-2} H_{00} y_k$ and hence $R^h_{0k} = -\frac{1}{(n-1)} H_{00} h^h_k$ which shows that the space is of scalar curvature. Now assume $W^{(2)'h}_{ijk} = 0$. Then

$$H^h_{ijk} + \frac{1}{(n+1)}\left(y^h \dot{\partial}_i H_{jk} + \frac{n+1}{n-1} \delta^h_j H_{ik} \right) = 0.$$

Contracting with respect to h and k, we get

$$H_{ij} + \frac{1}{(n+1)} \left\{ y^k \left(\dot{\partial}_i H_{jk} - \dot{\partial}_i H_{kj} \right) + \frac{(n+1)}{(n-1)} \left(\delta^k_j H_{ik} - n H_{ij} \right) \right\} = 0$$

or

$$y^k \left(\dot{\partial}_i H_{jk} - \dot{\partial}_i H_{kj} \right) = 0$$

i.e. $G_{ij(0)} = 0$, $G_{ij} := G^m_{ijm}$ (hv-Ricci tensor)

showing that F^n is of constant curvature (cf. references (1.11), Lemma and Theorem 1 of Fukui and Yamada[2]).

Conversly if K is constant, then from (3.3), (3.4) and (3.5), we have

$$W^{(2)'h}_{ijk} = K g_{ij} \delta^h_k + \frac{1}{n+1} \left\{ y^h (n-1)\, 2K C_{ijk} + (n+1)\, K \delta^h_j g_{ik} \right\} - j \mid k \qquad \ldots(4.2)$$

$$= K g_{ij} \delta^h_k + K g_{ik} \delta^h_j - j \mid k = 0.$$

Next, we shall prove that

Theorem 4.2—$W^{(2)'h}_{ijk}$ is invariant under a C-projective mapping.

PROOF : We know that

$$\overline{W}^{(2)'h}_{ijk} = \overline{H}^h_{ijk} + \frac{1}{(n+1)} \left\{ y^h \dot{\partial}_i \overline{H}_{jk} + \frac{(n+1)}{(n-1)} \delta^h_j \overline{H}_{ik} - j \mid k \right\}. \qquad \ldots(4.3)$$

Using (3.12), (3.13) and (3.14), we have

$$\overline{W}^{(2)'h}_{ijk} = H^h_{ijk} + \delta^h_j \dot{\partial}_i Q_k - \delta^h_k \dot{\partial}_i Q_j + \frac{1}{(n+1)} \Big[\Big\{ y^h \left(\dot{\partial}_i H_{jk} + \dot{\partial}_i \left(\dot{\partial}_j Q_k - n \dot{\partial}_j Q_k \right) \right\} + \frac{(n+1)}{(n-1)} \delta^h_j \left(H_{ik} - (n-1) \dot{\partial}_i Q_k \right) - j \mid k \Big]$$

(*equation continued on p.* 72)

$$= W^{(2)'h}_{ijk} + \delta^h_j \dot{\partial}_l Q_k - \delta^h_k \dot{\partial}_l Q_j + \frac{1}{(n+1)}\Big\{\Big[- (n-1) y^h$$

$$\dot{\partial}_l \dot{\partial}_j Q_k - (n+1) \delta^h_j \dot{\partial}_l Q_k\Big] - j \mid k \Big\}$$

$$= W^{(2)'h}_{ijk} - \frac{(n-1)}{(n+1)} y^h \dot{\partial}_l \left(\dot{\partial}_j Q_k - \dot{\partial}_k Q_j\right)$$

$$= W^{(2)'h}_{ijk}.$$

Theorem 4.3—A Finsler space F^n $(n \geqslant 3)$ of scalar curvature turns into a space of constant curvature if

$$\dot{\partial}_k (p . (\dot{\partial}_l H_{0l})) - j \mid k = 0. \qquad ...(4.4)$$

PROOF : From (3. 4), We have

$$\dot{\partial}_j H_{0l} = (n-1) K g_{lj} + \frac{1}{3}\{(n-2)(K_{lj} + K_l l_j) + (2n-1) K_j l_l\}. \qquad ...(4.5)$$

Taking indicatric projection $p.$ in (4.5), we have

$$p. (\dot{\partial}_j H_{0l}) = (n-1) K h_{lj} + \frac{(n-2)}{3} K_{lj}. \qquad ...(4.6)$$

Differentiating (4.6) partially with respet to y^k, we have

$$L \dot{\partial}_k (p . (\dot{\partial}_j H_{0l}) = (n-1)[K_k h_{lj} + 2LK C_{ijk} - h_{lk} l_j$$

$$- h_{jk} l_l] + \tfrac{1}{3}(n-2)(K_{klj} - l_l K_{kj} - l_j K_{kj}) \qquad ...(4.7)$$

where

$$K_{khj} := L p . (\dot{\partial}_k K_{hj}) = L \dot{\partial}_k K_{hj} + l_h K_{kj} + l_j K_{kh}.$$

Taking the skew-symmetric part of (4.7) in the indices j, k and using the condition (4.4), we have

$$(n-1)(K_k h_{lj} - h_{lk} l_j) + \tfrac{1}{3}(n-2)[K_{klj} - l_j K_{kl}] - j \mid k = 0 \qquad ...(4.8)$$

whose indicatric projection $p.$ yields

$$(n-1) K_k h_{lj} + \tfrac{1}{3}(n-2) K_{klj} - j \mid k = 0. \qquad ...(4.9)$$

Lemma [ref. 4, (3.18)]—In Finsler space of scalar curvature, the following relation holds

$$K_{ijk} + K_i h_{jk} - i \mid j = 0. \qquad \text{.. (4.10)}$$

Using (4.10) in (4.9), we have

$$(n-1) K_k h_{ij} + \tfrac{1}{3}(n-2) K_k h_{ij} - j \mid k = 0$$

i. e.

$$K_k h_{ij} - j \mid k = 0. \qquad \text{...(4.11)}$$

Transvecting (4.11) by h^{ij} we get $K_k = 0$. From (3.5) we have the assertion.

Theorem 4.4—A Finsler space F^n $(n \geqslant 3)$ of scalar curvature turns into a space of constant curvature if and only if

$$p \,.\, [\dot{\partial}_k (p \,.\, (\dot{\partial}_j H_{0l}))] - j \mid k = 0. \qquad \text{...(4.12)}$$

PROOF : Taking indicatric projection p . in (4.7) and skew-symmetry in j, k, we have

$$L\, p \,.\, [\dot{\partial}_k (p \,.\, (\dot{\partial}_j H_{0l}))] - j \mid k$$

$$= (n-1)(K_k h_{lj}) + \tfrac{1}{3}(n-2) K_{klj} - j \mid k. \qquad \text{...(4.13)}$$

Suppose that F^n is of constant curvature then $K_l = 0 = K_{lj0}$ (Sakaguchi[4], Theorem 4.5) and thus (4.12) holds.

Conversely if (4.12) holds then (4.13) implies (4.9) and then retreating the steps we have the assertion.

ACKNOWLEDGEMENT

The second author is thankful to C.S.I.R. for financial support.

REFERENCES

1. L. Berwald, *Ann. Math.* **48** (1947), 755-81.
2. M. Fukui, and T. Yamada, *Tensor, N. S.* **38** (1982), 129-34.
3. M. Fukui, and T. Yamada, *Tensor, N. S.* **35** (1981), 216-22.
4. T. Sakaguchi, *Tensor, N.S.* **38** (1982), 211-19.
5. C. Shibata, *Tensor, N. S.* **32** (1978), 311-17.
6. B. B. Sinha, Differential Geometry of Finsler Bundle, Banaras Hindu University 1981.