*Indian J. pure appl. Math.*, 17 (8) : 1008-1013, August 1986

# LIMIT CLASSIFICATION OF A REAL QUADRATIC FORM OF AN ORDINARY SECOND-ORDER SELF ADJOINT DIFFERENTIAL EXPRESSION

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(*Received* 21 *January* 1985; *after revision* 31 *March* 1986)

A classification of differential expressions $Ly = -y(2) + qy$, depending on the number of square-integrable solutions of $Ly = \lambda y$ for $\text{im}\,\lambda \neq 0$, is well-known. Similar classification of the differential expressions $L^2y = L(Ly)$ can be obtained having the knowledge of the classification of $L$, Chaudhary and Everitt[6]. Here the classification of $aL^2 + bL + c$, where $a, b, c$ are real constants, is determined, the classification of $L$ being known.

§ 1. We consider the self adjoint differential expression $L[y] = -y^{(2)}(x) + (qx)\,y(x)$, $0 \leqslant x < \infty$, where $q(.)$ is real valued, $y^{(r)}(x) \equiv \frac{d^r y}{dx^r}$. The differential expression $L[.]$ is said to be in the limit point $(L-P)$ case at infinity if there is at least one linearly independent solution not belonging to the space $\mathscr{L}^2 \equiv \mathscr{L}^2[0, \infty)$, the space of all Lebesque square integrable functions, of the differential equation (D.E.) $L[y] = \lambda y$, where $\lambda$ is a complex parameter and $\text{im}\,\lambda \neq 0$. If both the solutions of $L[y] = \lambda y$ are in $\mathscr{L}^2$, $L[.]$ is said to be in the limit-circle $(L-C)$ case at infinity. A self adjoint fourth order differential expression $M[.]$ is said to be in the limit $-r$ $(L-r)$ case at infinity if the number of linearly independent solutions of the D.E. $M[y] = \lambda y$, $\text{im}\,\lambda \neq 0$ is $r, r = 2, 3, 4$ [vide Titchmarsh[1], Everitt[2]]. The object of this note is to derive the limit-classification of $M[.] = a\,L^2[.] + b\,L[.] + c$ (where $a, b, c$ are real constants) from the knowledge of the limit classification of $L[.]$. The case $N[.] = L^2[.] + L[.]$ is considerd in detail; the argument for the general quadratic, with real coefficients, follows from this special case.

§ 2. It can be easily verified that

$$L^2[y] \equiv D^4y - D(2qD)y + (q^2 - D^2 q)y, \text{ provided } q,\ q \in AC_{\text{loc}}[0, \infty)$$

$$\left[D \equiv \frac{d}{dx^r}\right]$$

and

$$N[y] \equiv (L^2 + L)[y] = D^4y - D(2q\,D + D)\,y + (q^2 - D^2 q + q)\,y.$$

From the Green's formula[3] (chapter 3) we get

$$\int_{x_1}^{x_2} (vL[u] - uL[v])\, dx = [uv](x_2) - [uv](x_1)$$

where

$$[uv](x) = u(x)\, v^{(1)}(x) - u^{(1)}(x)\, v(x)$$

is called the bilinear form related to $L[.]$; we denote it by $[uv]_L$. Similarly it can be proved that the bilinear forms related to $L^2[.]$ and $N[.]$ are respectively

$$[uv]_{L^2} = u^{(3)} v - u^{(2)} v^{(1)} + u^{(1)} v^{(2)} - uv^{(3)} + 2q(.)(u v^{(1)} - u^{(1)} v)$$

and

$$[uv]_N = u^{(3)} v - u^{(2)} v^{(1)} + u^{(1)} v^{(2)} - uv^{(3)} + (2q(.)+1)(uv^{(1)} - u^{(1)} v).$$

It can be easily seen that

$$[uv]_N(x) = [uv]_L(x) + [uv]_{L^2}(x). \qquad \ldots(2.1)$$

§ 3. We define three operators $T_i : D(T_i) \subset \mathscr{L}^2 \to \mathscr{L}^2$, $(i = 1, 2, 3)$ as follows:

$T_1 : T_1 f(.) = L[f(.)]$, for all $f(.) \in D(T_1)$

$T_2 : T_2 f(.) = L^2[f(.)]$, for all $f(.) \in D(T_2)$

$T_3 : T_3 f(.) = N[f(.)]$, for all $f(.) \in D(T_3)$;

for the definitions of $D(T_i)$, $i = 1, 2, 3$, we first make a list of conditions on $f(.)$ :

(i) $f(.) \in \mathscr{L}^2$

(ii) $f^{(1)}(.)$ is absolutely continuous on $[0, X]$ for all $X > 0$

(iii) $L[f(.)] \in \mathscr{L}^2$

(iv) $L[f(.)]^{(1)}$ is absolutely continuous on $[0, X]$ for all $X > 0$

(v) $L^2[f(.)] \in \mathscr{L}^2$

then $f(.) \in D(T_1)$ if $f(.)$ satisfies (i), (ii) and (iii);

$f(.) \in D(T_2)$ if $f(.)$ satisfies (i), (ii), (iv) and (v)

$f(.) \in D(T_3)$ if $f(.)$ satisfies (i), (ii), (iii), (iv) and (v).

§ 4. It is known that [Everitt[4], § 8; Chaudhary and Everitt[6] § 9] a necessary and sufficient condition for a 2nd order differential expression $L[.]$ to be in the $L$-$P$ case at infinity is

$$[f, g]_L(x) \to 0 \text{ as } x \to \infty$$

for all

$f(.), g(.)$ satisfiying (i), (ii) and (ii) of Section 3. ...(4.1)

and that for a fourth order differential expression $N[.]$ to be in $L-2$ case at infinity is

$$[fg]_N(x) \to 0 \text{ as } x \to \infty$$

for all

$f(.), g(.)$ satisfying (i), (ii), (iii) and (v) of Section 3. ...(4.2)

It is to be noted that $L(L+1)L[y] \equiv (L+1)L[y]$. Hence if $\phi$ is a solution of $L[y] = \lambda y$, then $(L^2+L)[\phi] = (L+1)[\lambda\phi] = \lambda(\lambda+1)\phi$.

Again, if $\psi$ is a solution of $(L+1)[y] = -\lambda y$, then

$$(L^2+L)[\psi] = L(L+1)[\psi] = -L[\lambda\psi] = \lambda(\lambda+1)\psi.$$

Thus the the solutions of

$$(L^2+L)[y] = (\lambda^2+\lambda)y \qquad ...(4.3)$$

are the solutions of

$$L[y] = \lambda y \qquad ...(4.4)$$

and

$$L[y] = -(\lambda+1)y. \qquad ...(4.5)$$

Let $\phi(x, \lambda)\ \theta(x, \lambda)$ be a pair of fundamental solutions of the D. E. (4.4). Then $\phi(x, -\lambda-1)$, $\theta(x, -\lambda-1)$ form a pair of fundamental solutions of the D. E. (4.5). If $\phi(x, \lambda)$, $\theta(x, \lambda)$, $\phi(x, -\lambda-1)$ and $\theta(x, -\lambda-1)$ are not linearly independent, then there exists $A, B, C, D$ not all zero, such that

$$A\phi(x, \lambda)+B\theta(x, \lambda)+C\phi(x,-\lambda-1)+D\theta(x,-\lambda-1) = 0 \text{ for all } x \in [0,\infty). \qquad ...(4.6)$$

The above relation being identically true in $[0, \infty)$, we have

$$L[A\phi(x, \lambda) + B\theta(x, \lambda) + C\phi(x,-\lambda-1) + D\theta(x,-\lambda-1)] = 0$$

or

$$A\phi(x, \lambda)+B\theta(x, \lambda) - (\lambda+1)C\phi(x,-\lambda-1) - (\lambda+1)D\theta(x,-\lambda-1) = 0 \qquad ...(4.7)$$

From (4.6) and (4.7), we have,

$$A\phi(x, \lambda) + B\theta(x, \lambda) = 0 = C\phi(x,-\lambda-1) + D\theta(x,-\lambda-1);$$

whence it follows, $A = B = C = D = 0$. Hence $\phi(x, \lambda)$, $\theta(x, \lambda)$, $\phi(x,-\lambda-1)$ and $\theta(x,-\lambda-1)$ are linearly independent.

§ 5. It is known that (Chaudhury and Everitt[6], § 7, § 8) if $L[.]$ is in the $L-p$ case at infinity then $L^2[.]$ is either in the $L-2$ case at infinity or in the $L-3$ case at infinity. But if $L[.]$ is in the $L-c$ case at infinity then $L^2[.]$ is in the $L-4$ case at infinity. We shall prove the following :

(a) when $L[.]$ is in the $L-P$ case and $L^2[.]$ is in the $L-2$ case at infinity then $N[.]$ is in the $L-2$ case at infinity;

(b) when $L[.]$ is in the $L-P$ case but $L^2[.]$ is in the $L-3$ case at infinity then $N[.]$ is in the $L-3$ case at infinity

(c) when $L[.]$ is in the $L-C$ case at infinity then $N[.]$ is in the $L-4$ case at infinity.

*Proof of (a)*

$L[.]$ is in the $L-P$ case at infinity implies

$$[fg]_L(x) \to 0 \text{ as } x \to \infty, \text{ for all } f(.), g(.) \in D(T_1). \qquad \ldots(5.1)$$

$L^2[.]$ is in the $L-2$ case at infinity implies

$$[fg]_{\underset{L}{2}}(x) \to 0 \text{ as } x \to \infty, \text{ for all } f(.), g(.) \in D(T_2). \qquad \ldots(5.2)$$

Further, as $L^2[.]$ is in the $L-2$ case at infinity, we have (by virtue of Theorem 2 of Chaudhury and Everitt[6])

$$f(.), L^2[f] \in \mathscr{L}^2 \text{ implies } L[f] \in \mathscr{L}^2.$$

Hence, in this case $D(T_3) = D(T_2) \subset D(T_1)$.

So, for $f(.), g(.) \in D(T_3)$,

$$[fg]_N(x) = [fg]_{\underset{L}{2}}(x) + [fg]_L(x) \to 0 \text{ as } x \to \infty.$$

This indicates that $N[.]$ is in the limit-2 case at infinity.

*Proof of (b)*

Suppose $L[.]$ is in the $L-P$ case at infinity but $L^2[.]$ is in the $L-3$ case at infinity. Now $N[.]$ may be in the $L-2$ case at infinity or in the $L-3$ case at infinity or in the $L-4$ case at infinity. In order to establish (b) we need only to show that in this case $N[.]$ is neither in the $L-2$ case at infinity nor in the $L-4$ case at infinity. Obviously $N[.]$ can not be in the $L-2$ case at infinity because if it is so then

$$[fg]_N(x) \to 0 \text{ as } x \to \infty \text{ for all } f(.), g(.) \in (T_3)$$

which implies

$$[fg]_{\underset{L}{2}}(x) + [fg]_L(x) \to 0 \text{ as } x \to \infty \text{ for all } f(.), g(.) \in D(T_3). \qquad \ldots(5.3)$$

$L[.]$ being in the $L-P$ case at infinity, $[fg]_L(x) \to 0$ as $x \to \infty$, for all $f(.), g(.) \in D(T_1)$ and so for all $f(.)\, g(.) \in D(T_3)$. Since $D(T_3)$ is dense in $D(T_2)$, it will then follow that $L^2[.]$ is in $L-2$ case at infinity. This is contradictory to our hypothesis that $L^2[.]$ is in the $L-3$ case at infinity. Hence $N[.]$ is not in the $L-2$ ease at infinity. Next, if possible, suppose that $N[.]$ is in the $L-4$ case at infinity. Then the D. E.

$$N[y] = -\frac{1}{2}\, y \qquad \ldots(5.4)$$

has four linearly independent $\mathscr{L}^2$ solutions, since $L-4$ classification is independent of the parameter[2]. Now according to the discussions in Section 4, $\phi\left(x, \frac{i-1}{2}\right)$, $\theta\left(x, \frac{-i-1}{2}\right)$, $\theta\left(\frac{-i-1}{2}\right)$, $\theta\left(x, \frac{i-1}{2}\right)$ are all solutions of the D. E. (5.4), and hence these are all members of the space $\mathscr{L}^2$. Noting that, among them are $\phi\left(x, \frac{i-1}{2}\right)$ and $\theta\left(\frac{i-1}{2}\right)$which are two linearly independent solutions of the D. E.

$$L[y] = \frac{i-1}{2}\, y. \qquad \ldots(5.5).$$

We find the that $L[.]$ is in the $L-C$ case at infinity, contrary to hypothesis. Therefore $N[.]$ is in the $L-3$ case at infinity.

*Proof of* (c)

L [.] is in the $L-C$ case at infinity implies that $L^2[.]$ is in the $L-4$ case at infinity (Chaudhry andEveritt[6], § 5]. Now $L[.]$ is in the $L-C$ case at infinity implies that each of the D. E.'s

$$L[y] = \frac{i-1}{2}\, y \qquad \ldots(5.6)$$

and

$$L[y] = \frac{-i-1}{2}\, y \qquad \ldots(5.7)$$

have two linearly independent $\mathscr{L}^2$ solutions. Clearly $\left\{\phi\left(x, \frac{-1+i}{2}\right), \theta\left(x, \frac{-1+i}{2}\right)\right\}$ and $\left\{\phi\left(x, \frac{-1-i}{2}\right), \theta\left(x, \frac{-1-i}{2}\right)\right\}$ from sets of linearly independent solutions of the differential equations of the (5.6) and (5.7) respectively. Again these four functions are linearly independent solutions of the D. E. (5.4) and this implies that $N[.]$ is in $L-4$ case at infinity.

§ 6. In case of the general quadratic given by $M[.]$ we simply note that the bilinear form relative to $M[.]$ is given by

$$[uv]_M = a\,(u^{(3)}\,v - u^{(2)}\,v^{(1)} + u^{(1)}\,v^{(2)} - uv^{(3)}) + b\,(2q+1)\,(uv^{(1)} - u^{(1)}\,v)$$

$$= a\,[uv]_{\underset{L}{2}} + b\,[uv]_L$$

As $a$ and $b$ are given constants, the above analysis holds verbatim for $M[.]$.

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