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WEBER TRANSFORMATION OF THE SECOND KIND

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It is shown that the Weber transformation of the second kind is a continuous linear mapping of a certain test function U_μ into Zemanian's space $H_{\mu-1/2}$.

1. INTRODUCTION

In this paper we consider the Weber transform of the second kind defined by

$$W[G(y)](x) := g(\lambda) = \int_0^\infty \frac{y\, C_\mu(xy,ay)}{Q_\mu^2(ay)}\, G(y)\, dy \qquad \ldots(1.1)$$

with inversion formula given by

$$W^{-1}[g(x)](y) := G(y) = \int_0^\infty x\, C_\mu(xy, ay)\, g(x)\, dx. \qquad \ldots(1.2)$$

Here we show that the Weber transform W as defined by (1.1) is a continuous linear mapping of a certain testing function space U_μ into Zemanian's space $H_{\mu-1/2}$.

We shall use the following two linear differential operators defined by

$$N_\mu \phi(x) := x^{\mu+1/2} D\, x^{-\mu-1/2} \phi(x)$$

and

$$M_\mu \phi(x) := x^{-\mu-1/2} D\, x^{\mu+1/2} \phi(x).$$

2. THE TESTING FUNCTION SPACE H_μ

A function $\phi(x)$ is in H_μ if and only if it is defined on $I = (0, \infty)$, it is complex-valued and smooth and for each pair of nonnegative integers m and k,

$$\gamma_{m,k}^{\mu}(\phi) := \sup_{0<x<\infty} |\, x^m (x^{-1} D)^k [x^{-\mu-1/2} \phi(x)]\, | \qquad \ldots(2.1)$$

is finite. H_μ is sequentially complete, Hausdorff, topological vector space. For its various properties we may refer to Zemanian[6]. The dual of the space H_μ will be denoted by H'_μ which is also complete.

3. The Testing Function Space U_μ

A complex-valued infinitely differentiable function ϕ defined over I is said to belong to U_μ if

$$\beta_k(\phi) = \beta^{\mu}_{m,k}(\phi) := \sup_{\substack{x \\ m \leqslant k}} \left| x^m (x^{-1} D)^k \frac{\phi(x)}{Q^2_\mu(ax)} \right| \qquad \ldots(3.1)$$

is finite, where m, k are nonnegative integers and

$$Q^2_\mu(ax) = J^2_\mu(ax) + Y^2_\mu(ax).$$

The topology in U_μ is generated by a sequence of seminorms $\{\beta_k\}_{k=0}^{\infty}$ where

$$\beta_k(\phi) = \gamma_k \left\{ \frac{\phi(k)}{\mu(x)} \right\} \qquad \ldots(3.2)$$

and

$$\mu(x) = x^{-\mu-1/2} Q^2_\mu(ax).$$

A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is said to converge *ih* U_μ to ϕ if $\gamma_k(\phi_\nu - \phi) \to 0$ as $\nu \to \infty$ for each $k = 0, 1, 2, \ldots$. A sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ is said to be a Cauchy sequence in U_μ if $\gamma_k(\phi_m - \phi_n) \to 0$ as m and n both tend to infinity independently of each other. U_μ is also sequentially complete, Hausdorff, locally convex, topological linear space. The proof of completeness of U_μ is similar to that of H_μ. The dual of the space U_μ will be represented by U'_μ.

4. Some Operations on U_μ and U'_μ

(i) *Multipliers in* U_μ —Let θ denote the linear space of all smooth functions $\theta(x)$ defined on I such that, for each nonnegative integer r, there exists an integer n_r for which

$$\frac{(x^{-1} D)^r \theta(x)}{1 + x^{n_r}}$$

is bounded on I.

We show that any $\theta \in \Theta$ is a multiplier for U_μ for every μ. For $\phi \in U_\mu$, we have

$$(x^{-1} D)^k [Q_\mu (ax)]^{-2} \theta \phi = \sum_{r=0}^{k} \binom{k}{r} \frac{(x^{-1} D)^r \theta}{1 + x^{n_r}} (1 + x^{n_r})$$

$$\times (x^{-1} D)^{k-r} [Q_\mu (ax)]^{-2} \phi.$$

So that

$$\beta^{\mu}_{m,k} (\theta \phi) \leqslant \sum_{k=0}^{k} \binom{k}{r} B_r \sup_x [x^m (x^{-1} D)^{k-r} (Q_\mu (ax))^{-2} \phi$$

$$+ x^{m+n_r} (x^{-1} D)^{k-r} (Q_\mu (ax))^{-2} \phi]$$

i. e.

$$\beta^{\mu}_{m,k} (\theta \phi) \leqslant \sum_{r=0}^{k} \binom{k}{r} \left[\beta_r \beta^{\mu}_{m,k-r} (\phi) + \beta^{\mu}_{m+n_r,k-r} (\phi) \right]$$

where B_r are constants. This proves that the linear operator $\phi \to \theta\phi$ is a continuous mapping of U_μ into itself.

The adjoint operator $f \to \theta f$ defined on U'_μ, by

$$\langle \theta f, \phi \rangle := \langle f, \theta\phi \rangle \quad f \in U'_\mu, \phi \in U_\mu, \theta \in \Theta \qquad \ldots(4.1)$$

is a continuous linear mapping of U'_μ into itself. Notice that the space Θ does not depend on the choice of μ.

(ii) If $P(x)$ and $Q(x)$ are polynomials and $Q(x)$ has no zeros on $0 \leqslant x < \infty$ then $P(x^2)/Q(x^2)$ is a member of Θ.

PROOF : See Zemanian[6] (p. 134).

(iii) For any integer n and for any μ, the mapping $\phi(x) \to x^n \phi(x)$ is an isomorphism from U_μ onto $U_{\mu+n}$. Therefore, the operator $f(x) \to x^n f(x)$, which is defined by

$$\langle x^n f(x), \phi(x) \rangle := \langle f(x), x^n \phi(x) \rangle \qquad \ldots(4.2)$$

is an isomorphism from $U'_{\mu+n}$ onto U'_μ

PROOF : If $\phi \in U_\mu$, then

$$\gamma^{\mu+n} (x^n \phi) = \gamma^{\mu}_{m,k} (\phi).$$

The second part follows on using Theorem 1.10 – 1 of Zemanian[6] (p. 29).

(ix) For every choice of the nonnegative integer r, let

$$\rho_r^{\mu}(\phi) \max_{\substack{0\leqslant m\leqslant r \\ 0\leqslant k\leqslant r}} \gamma_{m,k}^{\mu}(\phi) . \qquad \ldots(4.3)$$

By Theorem 1.8 – 1 of Zemanian[6] (p. 18 – 19) for each $f \in U_{\mu}'$ there exist a positive constant C and a nonnegative integer r such that

$$\left| \langle f, \phi \rangle \right| \leqslant C\rho_r^{\mu}(\phi)$$

for every $\phi \in U_{\mu}$. Here, C and r depend on f but not on ϕ.

5. Weber Transform of the Second Kind on U_{μ} Space

Theorem 5.1—The Weber transform W is a continuous linear mapping from the space U_{μ} in to $H_{\mu-1/2}$.

Proof : Let $\phi(x) \in U_{\mu}$ and let $\Phi(y)$ denote its Weber transform. Then

$$\Phi(y) = \int_0^{\infty} \phi(x) \frac{x}{Q_{\mu}^2(ax)} C_{\mu}(xy, ax)\, dx$$

$$= \int_0^{\infty} \phi(x) \frac{x}{Q_{\mu}^2(ax)} [J_{\mu}(xy)\, Y_{\mu}(ax) - J_{\mu}(ax)\, Y_{\mu}(xy)]\, dx$$

$$= \int_0^{\infty} \phi(x) \frac{x}{Q_{\mu}^2(ax)} Y_{\mu}(ax)\, J_{\mu}(xy)\, dx$$

$$- \int_0^{\infty} \phi(x) \frac{x}{Q_{\mu}^2(ax)} J_{\mu}(ax)\, Y_{\mu}(xy)\, dx.$$

Now, writing $\Phi^*(y)$ for $y^{1/2}\Phi(y)$, we have

$$\Phi^*(y) = y^{1/2}\Phi(y) = \int_0^{\infty} \phi(x) \frac{x^{1/2}}{Q_{\mu}^2(ax)} Y_{\mu}(ax)\, ((xy)^{1/2} J_{\mu}(xy))\, dx$$

(*equation continued on p.* 1018)

$$- \int_0^\infty \phi\ (x)\ \frac{x^{1/2}}{Q_\mu^2\ (ax)}\ J_\mu\ (ax)\ ((xy)^{1/2}\ Y_\mu\ (xy))\ dx.$$

Therefore,

$$\Phi^*\ (y) = \int_0^\infty \phi_1\ (x)\ ((xy)^{1/2}\ J_\mu\ (xy))\ dx\ - \int_0^\infty \phi_2\ (x)\ ((xy)^{1/2}\ Y_\mu\ (xy))\ dx$$

where

$$\phi_1\ (x) : =\ \frac{x^{1/2}}{Q_\mu^2\ (ax)}\ Y_\mu\ (ax)\ \phi\ (x)$$

and

$$\phi_2\ (x) : =\ \frac{x^{1/2}}{Q_\mu^2\ (ax)}\ J_\mu\ (ax)\ \phi\ (x).$$

Therefore

$$\Phi^*\ (y) = h_\mu\ \phi_1\ \mathscr{Y}_\mu\ \phi_2 \qquad \ldots(5.1)$$

where

$$h_\mu\ [\phi_1]\ (y) = \int_0^\infty \phi_1\ (x)\ (xy)^{1/2}\ J_\mu\ (xy)\ dx$$

and

$$\mathscr{Y}_\mu\ [\phi_2]\ (y)\ = \int_0^\infty \phi_2\ (x)\ (xy)^{1/2}\ Y_\mu\ (xy)\ dx.$$

Now,

$$h_{\mu+1}\ N_\mu\ \phi_1\ (x) = \int_0^\infty\ [N_\mu\ \phi_1\ (x)\ ]\ (xy)^{1/2}\ J_{\mu+1}\ (xy)\ dx$$

$$= \int_0^\infty (xy)^{1/2}\ J_{\mu+1}\ (xy)\ x^{\mu+1/2}\ D_x\ x^{-\mu-1/2}\ \phi_1\ (x)\ dx$$

$$= (xy)^{1/2}\ J_{\mu+1}\ (xy)\ x^{\mu+1/2}\ \{x^{-\mu-1/2}\ \phi_1\ (x)\}\ \Big|_0^\infty$$

(equation continued on p. 1019)

$$-\int_0^\infty \frac{d}{dx}\ [(xy)^{1/2}\ J_{\mu+1}\ (xy)\ x^{\mu+1/2}]\ \ x^{-\mu-1/2}\ \phi_1\ (x)\ dx$$

$$= (xy)^{1/2}\ J_{\mu+1}\ (xy)\ \phi_1\ (x)\ \Big|_0^\infty - \int_0^\infty D_x\ [(y)^{1/2}\ x^{\mu+1}\ J_{\mu+1}\ (xy)]$$

$$\times\ x^{-\mu-1/2}$$

$$= (xy)^{1/2}\ J_{\mu+1}\ (xy)\ \phi_1\ (x)\ \Big|_0^\infty - \int_0^\infty y^{1/2}.\ y\ x^{\mu+1}\ J_\mu\ (xy)\ \ x^{-\mu-1/2}$$

$$\times\ \phi_1\ (x)\ dx$$

$$= (xy)^{1/2}\ J_{\mu+1}\ \ (xy)\ \ \frac{x^{1/2}}{Q_\mu^2\ \ (ax)}\ Y_\mu\ (ax)\ \phi\ (x)\ \Big|_0^\infty$$

$$-\ y\int_0^\infty (xy)^{1/2}\ J_\mu\ (xy)\ \phi_1\ (x)\ dx$$

$$= x\ y^{1/2}\ J_{\mu+1}\ (xy)\ \ \frac{Y_\mu\ (ax)}{Q_\mu^2\ (ax)}\ \phi\ (x)\ \Big|_0^\infty - y\ h_\mu\ \phi_1\ (x)$$

$$= -\ y\ h_\mu\ \phi_1\ (x) \qquad \ldots(5.2)$$

since the limit term vanishes.

Again,

$$\mathscr{J}_{\mu+1}\ (N_\mu\ \phi_2) = \int_0^\infty (xy)^{1/2}\ Y_{\mu+1}\ (xy)\ \ x^{\mu+1/2}\ D_x\ x^{-\mu-1/2}\ \phi_2\ (x)\ dx$$

$$= (xy)^{1/2}\ Y_{\mu+1}\ (xy)\ x^{\mu+1/2}.\ \ x^{-\mu-1/2}\ \phi_2\ (x)\ \Big|_0^\infty$$

$$-\int_0^\infty \frac{d}{dx}\ [(xy)^{1/2}\ Y_{\mu+1}\ (xy)\ x^{\mu+1/2}]\ x^{-\mu-1/2}\ \phi_2\ (x)\ dx$$

$$= (xy)^{1/2}\ Y_{\mu+1}\ (xy)\ \ \frac{x^{1/2}}{Q_\mu^2\ (ax)}\ \ J_\mu\ \ (ax)\ \phi\ (x)\ \Big|_0^\infty$$

(equation continued on p. 1020)

$$- \int_0^\infty y^{1/2}.y\, x^{\mu+1}\, Y_\mu\,(xy)\, x^{-\mu-1/2}\, \phi_2\,(x)\, dx$$

$$= x\, y^{1/2}\, Y_{\mu+1}\,(xy)\, J_\mu\,(ax)\, \frac{\phi(x)}{Q_\mu^2\,(ax)} \Bigg|_0^\infty$$

$$- y \int_0^\infty (xy)^{1/2}\, Y_\mu\,(xy)\, \phi_2\,(x)\, dx$$

$$= - y\, \mathscr{J}_\mu\, \phi_2 \qquad \ldots(5.3)$$

provided that

$$\lim_{x \to 0+} \frac{\phi(x)}{Q\,(ax)} = 0.$$

From (5.1), (5.2) and (5.3), we have

$$y\, \Phi^*\,(y) = y\,[h_\mu\, \phi_1 - \mathscr{J}_\mu\, \phi_2]$$

$$= - h_{\mu+1}\,(N_\mu\, \phi_1) + \mathscr{J}_{\mu+1}\,(N_\mu\, \phi_2). \qquad \ldots(5.4)$$

Applying the operator N_μ to (5.1), we have

$$N_\mu\, \Phi^*\,(y) = N_\mu\, h_\mu\, \phi_1 - N_\mu\, \mathscr{J}_\mu\, \phi_2. \qquad \ldots(5.5)$$

Now

$$N_\mu h_\mu\, \phi_1 = y^{\mu+1/2}\, D_y\, y^{-\mu-1/2} \int_0^\infty \phi\,(x)\, \frac{x^{1/2}}{Q_\mu^2\,(ax)}\, Y_\mu\,(ax)\,(xy)^{1/2}$$

$$\times\, J_\mu\,(xy)\, dx$$

$$= \int_0^\infty \phi\,(x)\, \frac{x}{Q_\mu^2\,(ax)}\, Y_\mu\,(ax)\, y^{\mu+1/2}\, D_y\, y^{-\mu}\, J_\mu\,(xy)\, dx$$

$$= \int_0^\infty \phi(x)\, \frac{x}{Q_\mu^2\,(ax)}\, Y_\mu\,(ax)\, y^{\mu+1/2}\,(-\,x)\, y^{-\mu}\, J_{\mu+1}\,(xy)\, dx$$

$$= \int_0^\infty \phi(x)\, \frac{x^{1/2}}{Q_\mu^2\,(ax)}\, Y_\mu\,(ax)\,(-\,x)\,(xy)^{1/2}\, J_{\mu+1}\,(xy)\, dx$$

$$= h_{\mu+1}\,(-\,x\, \phi_1) \qquad \ldots(5.6)$$

and

$$N_\mu \mathscr{J}_\mu \phi_2 = y^{\mu+1/2} D_y y^{-\mu-1/2} \int_0^\infty \phi(x) \frac{x^{1/2}}{Q_\mu^2(ax)} J_\mu(ax) (xy)^{1/2} \times Y_\mu(xy)\, dx$$

$$= \int_0^\infty \phi(x) \frac{x}{Q_\mu^2(ax)} J_\mu(ax)\, y^{\mu+1/2} D_y y^{-\mu} Y_\mu(xy)\, dx$$

$$= \int_0^\infty \phi(x) \frac{x}{Q_\mu^2(ax)} J_\mu(ax)\, y^{\mu+1/2} (-xy^{-\mu}) Y_{\mu+1}(xy)\, dx$$

$$= \int_0^\infty \phi(x) \frac{x^{1/2}}{Q_\mu^2(ax)} J_\mu(ax) (-x) (xy)^{1/2} Y_{\mu+1}(xy)\, dx$$

$$= \mathscr{J}_{\mu+1}(-x\phi_2). \qquad \ldots(5.7)$$

From (5.5), (5.6) and (5.7), we get

$$N_\mu \Phi^*(y) = h_{\mu+1}(-x\phi_1) + \mathscr{J}_{\mu+1}(-x\phi_2). \qquad \ldots(5.8)$$

Applying the operator $N_{\mu+1}$ to (5.8), we obtain

$$N_{\mu+1} N_\mu \Phi^*(y) = N_{\mu+1}[h_{\mu+1}(-x\phi_1) + \mathscr{J}_{\mu+1}(-x\phi_2)]$$
$$= h_{\mu+2}[(-x)(-x\phi_1)] + \mathscr{J}_{\mu+2}[(-x)(-x\phi_2)]$$
$$= h_{\mu+2}[(-x)^2\phi_1] + \mathscr{J}_{\mu+2}[(-x)^2\phi_2].$$

Similarly, we have

$$N_{\mu+2} N_{\mu+1} N_\mu \Phi^*(y) = h_{\mu+3}[(-x)^3\phi_1] + \mathscr{J}_{\mu+3}[(-x)^3\phi_2].$$

Continuing this process k times where k is a nonnegative integer, we obtain

$$N_{\mu+k-1} \ldots N_{\mu+1} N_\mu \Phi^*(y) = h_{\mu+k}[(-x)^k\phi_1] + \mathscr{J}_{\mu+k}[(-x)^k\phi_2].$$

Multiplying both sides by $-y$, we have

$$(-y) N_{\mu+k-1} \ldots N_\mu \Phi^*(y) = (-y) h_{\mu+k}[(-x)^k\phi_1] + (-)\mathscr{J}_{\mu+k}[(-x)^k\phi_2].$$

Using (5.2) and (5.3) the right-hand side transforms into

$$h_{\mu+k+1}[N_{\mu+k}(-x)^k\phi_1] + \mathscr{J}_{\mu+k+1}[N_{\mu+k}(-x)^k\phi_2].$$

Again multiplying by $(-y)$, we have

$$(-y)^2 [N_{\mu+k+1} \dots N_\mu \Phi^*(y)] = (-y)[h_{\mu+k+1} N_{\mu+k}((-x)^k \phi_1)] + (-y)[\mathscr{J}_{\mu+k+1} N_{\mu+k}((-x)^k \phi_2)]$$
$$= h_{\mu+k+2} N_{\mu+k+1} N_{\mu+k}[(-x)^k \phi_1] + \mathscr{J}_{\mu+k+2} \times N_{\mu+k+1} N_{\mu+k}[(-x)^k \phi_2].$$

Continuing this process m times where m is a nonegative integer, we have

$$(-y)^m [N_{\mu+k-1} \dots N_\mu \Phi^*(y)] = h_{\mu+k+m}(N_{\mu+k+m-1} \dots N_{\mu+k}[(-x)^k \phi_1]) + \mathscr{J}_{\mu+k+m}(N_{\mu+k+m-1} \dots N_{\mu+k}[(-x)^k \phi_2]). \quad \dots(5.9)$$

Using the relation [Zemanian[6], p. 141].

$$N_{\mu+k+m-1} \dots N_{\mu+k} x^k \phi(x) = x^k N_{\mu+m-1} \dots N_\mu \phi(x)$$

the relation (5.9) reduces to

$$(-y)^m [N_{\mu+k-1} \dots N_\mu \Phi^*(y)]$$
$$= \int_0^\infty (-x)^k [N_{\mu+m-1} \dots N_\mu \phi_1(x)] (xy)^{1/2} J_{\mu+k+m}(xy)\, dx$$
$$+ \int_0^\infty (-x)^k [N_{\mu+m-1} \dots N_\mu \phi_2(x)] (xy)^{1/2} Y_{\mu+k+m}(xy)\, dx.$$

This is the same as

$$(-1)^{m+k} y^m (y^{-1} D_y)^k y^{-\mu} \Phi(y)$$
$$= \int_0^\infty x^{2\mu+2k+m-1} \left[(x^{-1} D_x)^m x^{-\mu} \frac{Y_\mu(ax)}{Q_\mu^2(ax)} \phi(x) \right] \times \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\, dx + \int_0^\infty x^{2\mu+2k+m+1} [(x^{-1} D_x)^m x^{-\mu} \times \frac{J_\mu(ax)}{Q_\mu^2(ax)} \phi(x)] \frac{Y_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\, dx$$
$$= I_1 + I_2 \text{ (say)}. \quad \dots(5.10)$$

$$I_1 = \int_0^\infty x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2(ax)} \right] \right.$$

(equation continued on p. 1023)

$$\times (x^{-1} D_x)^r [x^{-\mu} Y_\mu (ax)]) \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}} dx$$

$$= \int_0^\infty x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2 (ax)} \right] (-a)^r \right.$$

$$\left. \times x^{-(\mu+r)} Y_{\mu+r} (ax) \right) \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}} dx.$$

For $y > a > 0$, we have

$$I_1 = \int_0^{1/a} \ldots dx + \int_{1/a}^\infty \ldots dx.$$

$$= I_3 + I_4 \text{ (say).}$$

Now,

$$I_3 = \int_0^{1/a} x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2 (ax)} \right] \right.$$

$$\times (-a)^r x^{-(\mu+r)} O((ax))^{-(\mu+r)}) \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+k}} dx$$

$$= O\left(\int_0^{1/a} \sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2 (ax)} \right] (-1)^r a^{-\mu} x^{m-2r+2k+1} \right]$$

$$\left. \times \frac{J_{\mu+k+m}(xy)}{(xy)^{\mu+t}} dx \right)$$

$$\leqslant A_1 \sum_{r=0}^{m} \binom{m}{r} \beta_{(m-2r)+2k+1, m-r}^{\mu} (\phi) \qquad \ldots(5.11)$$

provided that $m \leqslant k$.

$$I_4 = \int_{1/a}^\infty x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2 (ax)} \right] \right.$$

$$\left. \times (-a)^r x^{-(\mu+r)} Y_{\mu+r} (ax)) \frac{J_{\mu+m+k}(xy)}{(xy)^{\mu+k}} dx \right.$$

$$= \int_{1/a}^\infty x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2 (ax)} \right] (-a)^r x^{-(\mu+r)} \right)$$

(*equation continued on p.* 1024)

$$O\left((ax)^{-1/2} \quad \frac{O\left((xy)^{-1/2}\right)}{(xy)^{\mu+k}}\, dx\right)$$

$$= y^{-\mu+k+1/2}\,(a)^{1/2}\, O\left(\int_0^{\infty} \sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r}\,(-a)^r \times \left[\frac{\phi(x)}{Q_\mu^2(ax)}\right] x^{k+m-r+2}\right)\frac{dx}{x^2}$$

$$= y^{-(\mu+k+1/2)} \quad a^{1/2}\; O\left(\sum_{r=0}^{m} \binom{m}{r} (-a)^r\, \beta^{\mu}_{k+m-r+2,m-r}(\phi) \int_{1/a}^{\infty} \frac{dx}{x^2}\right)$$

$$\leqslant A_1 \sum_{r=0} \binom{m}{r} \beta^{\mu}_{k+m-r+2,m-r} \quad (\phi). \qquad \ldots(5.12)$$

$$I_2 = \int_0^{\infty} x^{2\mu+2k+m+1} \left[(x^{-1} D_x)^m\, x^{-\mu} J_\mu(ax)\, \frac{\phi(x)}{Q_\mu^2(ax)}\right] \times \frac{Y_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\, dx$$

$$= \int_0^{\infty} x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r}\left[\frac{\phi(x)}{Q_\mu^2(ax)}\right] \times (x^{-1} D_x)^r\,[x^{-\mu} J_\mu(ax)]\, \frac{Y_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\, dx\right)$$

$$= \int_0^{\infty} x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r}\left[\frac{\phi(x)}{Q_\mu^2(ax)}\right] \times (-a^r)\, x^{-(\mu+r)} J_{\mu+r}(ax)\right) \frac{Y_{\mu+k+m}(xy)}{(xy)^{\mu+k}}\, dx$$

Let $y > a > 0$, then we have

$$I_2 = \int_0^{1/a} \ldots dx + \int_{1/a}^{\infty} \ldots dx$$

$$= I_5 + I_6 \text{ (say)}.$$

Then

$$I_5 = \int_0^{1/a} x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1}D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2(ax)} \right] (-a)^r \right.$$

$$\left. \times x^{-(\mu+r)} O[(ax)^{\mu+r}] \right) \frac{O(xy)^{-((\mu+r+m)}}{(xy)^{\mu+k}} dx$$

$$= O\left(y^{-m} \int_0^{1/a} \sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2(ax)} \right] (-a)^r x dx \right)$$

$$= O\left(y^{-m} \int_0^{1/a} \sum_{r=0}^{m} \binom{m}{r} x (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2(ax)} \right] dx \right)$$

$$\leqslant A_3 \sum_{r=0}^{m} \binom{m}{r} \beta^{\mu}_{1,m-r} \quad (\phi). \qquad \ldots(5.13)$$

$$I_6 = \int_{1/a}^{\infty} x^{2\mu+2k+m+1} \left(\sum_{r=0}^{m} \binom{m}{r} (x^{-1} D_x)^{m-r} \left[\frac{\phi(x)}{Q_\mu^2(ax)} \right] (-a)^r \right.$$

$$\left. \times x^{-(\mu+r)} \{O[(ax)^{-1/2}] O[xy)^{1/2}]/(xy)^{\mu+k} dx \right.$$

$$= O\left(\int_{1/a}^{\infty} y^{-(\mu+k+1/2)} \sum_{r=0}^{m} \binom{m}{r} x^{k+m-r+2} (x^{-1} D_x)^{m-r} \right.$$

$$\left. \times \left[\frac{\phi(x)}{Q_\mu^2(ax)} \right] (-a)^r a^{-1/2} \frac{dx}{x^2} \right)$$

$$< A_4 \sum_{r=0}^{m} \binom{m}{r} \beta^{\mu}_{k+m-r+2,m-r} \quad (\phi). \qquad \ldots(5.14)$$

Combining the results (5.10), (5.11), (5.12), (5.13) and (5.14), we get

$$(-1)^{m+k} y^m (y^{-1} D_y)^k y^{-\mu} \Phi(y) \leqslant \sum_{r=0}^{m} \binom{m}{r} A_1 \left[\beta\mu_{(m+2r)+2k+1,m-r}(\phi) \right.$$

$$+ A_2 \beta^{\mu}_{k+m-r+2,m-r}(\phi) + A_3 \beta^{\mu}{}_{1,m-r}(\phi)$$

$$\left. + A_4 \beta^{\mu}_{k+m-r+2,m-r}(\phi) \right].$$

Thus there exists a continuous linear mapping from the space U_μ into the space $H_{\mu-1/2}$

An Open Problem

So far we have not succeeded in showing that the inverse Weber transform W^{-1} is a continuous linear mapping from $H_{\mu-1/2}$ into U_μ.

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