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THE MEAN VALUES OF AN ANALYTIC FUNCTION REPRESENTED BY DIRICHLET SERIES II

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Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, $s = \sigma + it$, $0 \leqslant \lambda_1 < \lambda_2 < \lambda_n \to \infty$, be analytic in the half plane $\sigma < A$. The growth properties of the mean $L(\sigma) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \log |f(\sigma + it)| \, dt$ have been obtained in this paper. Theorem 1 generalizes and improves upon a result of J. S. Gupta and S. Bala which was proved for entire Dirichlet series.

§ 1. Let $f(s)$ be an analytic function in the half plane $\sigma < A$, where A is the abscissa of absolute convergence of the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n) \qquad \ldots (1.1)$$

$0 \leqslant \lambda_1 < \lambda_2 < \ldots$, $\lambda_n \to \infty$ as $n \to \infty$, $s = \sigma + it$ and

$$\lim_{n\to\infty} \sup \frac{n}{\lambda_n} = D < \infty.$$

The analytic function $f(s)$ is said to be of order ρ and lower order λ if

$$\lim_{\sigma\to A} \frac{\sup}{\inf} \frac{\log \log M(\sigma)}{\log x} = \frac{\rho}{\lambda}, \; 0 \leqslant \lambda \leqslant \rho \leqslant \infty. \qquad \ldots (1.2)$$

Moreover, for $0 < \rho < \infty$, type T and lower type t of $f(s)$ are defined as

$$\lim_{\sigma\to A} \frac{\sup}{\inf} \frac{\log M(\sigma)}{x^{\rho}} = \frac{T}{t}, \; 0 \leqslant t \leqslant T \leqslant \infty \qquad \ldots (1.3)$$

where

$$M(\sigma) = \text{l. u. b.} \quad |f(\sigma + it)| \text{ and } x = e^{\sigma}/(1 - e^{\sigma - A}).$$
$$-\infty < t < \infty.$$

For further details and properties of analytic functions, we shall refer to Juneja and Nandan[2].

For any $\sigma < A$, the logarithmic mean of $f(s)$ is defined as

$$L(\sigma) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \log |f(\sigma + it)| \, dt. \qquad \ldots(1.4)$$

Further, for any k, $-1 < k < \infty$, the generalized logarithmic mean function $G(\sigma)$ of $f(s)$ is defined as[3] :

$$G(\sigma) = \frac{1}{x^k} \int_{a}^{\sigma} \frac{e^{ky} L(y)}{(1 - e^{y - A})^{k+1}} \, dy \qquad \ldots(1.5)$$

where a is any fixed number such that $-\infty < a < \sigma < A$.

It is known[3] (p. 273) that

$$\lim_{\sigma \to A} \sup_{\inf} \frac{\log G(\sigma)}{\log x} \leqslant \frac{\rho}{\lambda}. \qquad \ldots(1.6)$$

In this paper, we continue our study of the growth properties of the means $G(\sigma)$ and $L(\sigma)$ that were initiated earlier[3]. One of our results not only generalizes but also improves upon the result of Gupta and Bala[1] (p. 7), proved for entire Dirichlet series.

§ 2. In this section we prove two theorems. Unless otherwise stated, we shall assume that $-\infty < A < \infty$ and $x = e^{\sigma}/(1 - e^{\sigma - A})$.

Theorem 1—Let $f(s)$ be an analytic function of order ρ and lower order λ. Then we have

$$\lim_{\sigma \to A} \sup_{\inf} \left[\frac{\sigma L(\sigma)}{G(\sigma) \log x} - k \right] \leqslant \frac{\rho}{\lambda}. \qquad \ldots(2.1)$$

PROOF : From (1.5) we have

$$\frac{d}{d\sigma} \{\log G(\sigma)\} = \frac{L(\sigma)}{(1 - e^{\sigma - A}) G(\sigma)} - \frac{k}{1 - e^{\sigma - A}}. \qquad \ldots(2.2)$$

Hence for $a < \sigma_0 < \sigma < A$, we get

$$\log G(\sigma) = \int_{\sigma_0}^{\sigma} \frac{L(y)}{(1 - e^{y-A})\, G(y)}\, dy - k \int_{\sigma_0}^{\sigma} \frac{dy}{1 - e^{y-A}} + O(1). \quad \ldots(2.3)$$

Let $\nu = \sigma(2 - e^{\sigma - A})$. Then $\nu > \sigma$ and $\lim_{\sigma \to A} \nu = A$. Thus from (2.3), since $\{L(\sigma)/(1 - e^{\sigma - A})\, G(\sigma)\}$ is an increasing function of σ [Kumar and Srivastava[3], Lemma 3], we have

$$\log G(\nu) > \frac{(\nu - \sigma)\, L(\sigma)}{(1 - e^{\sigma - A})\, G(\sigma)} + k \log\left(\frac{A - \nu}{A - \sigma_0}\right) + O(1)$$

or,

$$\frac{\log G(\nu)}{\log [e^{\nu}/(1 - e^{\nu - A})]} > G(\sigma) \frac{\sigma L(\sigma)}{\log [e^{\nu}/(1 - e^{\nu - A})]} + k \frac{\log (A - \nu)}{\log [e^{\nu}/(1 - e^{\nu - A})]} + o(1).$$

Since

$$\lim_{\sigma \to A} \frac{-\log (A - \nu)}{\log [e^{\nu}/(1 - e^{\nu - A})]} = \lim_{\sigma \to A} \frac{\log x}{\log [e^{\nu}/(1 - e^{\nu - A})]} = 1$$

taking limits in the last inequality we obtain

$$\lim_{\sigma \to A} \sup_{\inf} \left[\frac{\sigma L(\sigma)}{G(\sigma) \log x} - k\right] \leqslant \lim_{\sigma \to A} \sup_{\inf} \frac{\log G(\sigma)}{\log x}$$

and (2.1) follows in view of (1.6). This proves Theorem 1.

Remark : It is well known that for $A = \infty$, (1.1) represents an entire function and mean defined by (1.5) takes the simple from

$$G(\sigma) = \frac{1}{e^{k\sigma}} \int_0^{\sigma} e^{ky}\, L(y)\, dy.$$

In this case (2.1) reduces to

$$\lim_{\sigma \to \infty} \sup_{\inf} \left\{\frac{L(\sigma)}{G(\sigma)} - k\right\} \leqslant \frac{\rho^*}{\lambda^*} \quad \ldots(2.4)$$

where ρ^* and λ^* denote the order and lower order respectively, of the entire function $f(s)$. We see that (2.4) improves upon the following result of Gupta and Bala[1] (Th. 4) :

$$\lim_{\sigma \to \infty} \sup_{\inf} \left\{\frac{L(\sigma)}{G(\sigma)} - k\right\} \leqslant \frac{2\rho^*}{2\lambda^*}.$$

Theorem 2--If T and t are as defined by (1.3), then we have

$$\lim_{\sigma \to A} \begin{matrix} \sup \\ \inf \end{matrix} \frac{(1 - e^{\sigma - A})\, G'(\sigma) + kG(\sigma)}{x^{\rho}} \leqslant \begin{matrix} T \\ t \end{matrix} \qquad \ldots(2.5)$$

where $G'(\sigma)$ denotes the derivative of $G(\sigma)$.

PROOF : For $a < \sigma_1, \sigma_2 < A$, we get from (1.5),

$$\left(\frac{e^{\sigma_2}}{1 - e^{\sigma_2 - A}}\right)^k G(\sigma_2) - \left(\frac{e^{\sigma_1}}{1 - e^{\sigma_1 - A}}\right)^k G(\sigma_1) = \int_{\sigma_1}^{\sigma_2} \frac{e^{ky}\, L(y)}{(1 - e^{y-A})^{k+1}}\, dy. \qquad \ldots(2.6)$$

Further, (1.4) implies that $L(\sigma) \leqslant \log M(\sigma)$. Hence (2.6) yields

$$k \left\{\left(\frac{e^{\sigma_2}}{1 - e^{\sigma_2 - A}}\right)^k G(\sigma_2) - \left(\frac{e^{\sigma_1}}{1 - e^{\sigma_1 - A}}\right) G(\sigma_1)\right\} \leqslant \left\{\left(\frac{e^{\sigma_2}}{1 - e^{\sigma_2 - A}}\right)^k - \left(\frac{e^{\sigma_1}}{1 - e^{\sigma_1 - A}}\right)^k\right\} \log M(\sigma_2).$$

Putting $\sigma_1 = \sigma$ and $\sigma_2 = u = A - \frac{1}{h}(A - \sigma)$, $h > 1$ in the above inequality, we obtain

$$k \left\{\left(\frac{e^{u}}{1 - e^{u - A}}\right)^k G(u) - x^k G(\sigma)\right\} \leqslant \left\{\left(\frac{e^{u}}{1 - e^{u - A}}\right)^k - x^k\right\} \log M(u).$$

Dividing by the coefficient of $\log M(u)$ on both sides and taking limits as $h \to 1$, we get

$$(1 - e^{\sigma - A})\, G'(\sigma) + kG(\sigma) \leqslant \log M(\sigma).$$

The inequalities (2.5) now follow on taking the limits after dividing by x^{ρ} and using (1.3). This proves Theorem 2.

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