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ASYMPTOTIC BEHAVIOUR OF SIMILAR PROFILES

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The equation studied is

$$u''' + \alpha\, u\, u'' + \beta\,(1 - u'^2) + \gamma\, u'' = 0.$$

Here $u = u(t)$, together with the boundary conditions $u(0) = u'(0) = 0$, $(\infty) = 1$. The value of α is considered to be unity, whereas $\gamma\,(-\infty < \gamma < (\infty)$ has been chosen arbitrarily. Along with the above boundary conditions, the side condition $0 < u' < 1$ on $(0, \infty)$ has also been taken into consideration. The present paper deals with asymptotic behaviour, as $t \to \infty$, of the solutions of the above non-linear, autonomous third order, differential equation. The results are based on the asymptotic integrations of the second order, linear differential equations.

INTRODUCTION

The Falkner-Skan equation has been the subject of considerable research in the last forty years due to its importance in the boundary layer theory. According to the assumptions of boundary layer theory[1] the steady two dimensional flow of a slightly viscous incompressible fluid past a wedge of angle $\pi\beta$ $(0 \leqslant \beta < 2)$ can be described in terms of the solution of the differential equation

$$u''' + uu'' + \beta\,(1 - u'^2) = 0 \qquad \ldots(1)$$

(where $u = u(x)$ and primes denote derivatives w. r. t. the variable x) which satisfies the boundary condition

$$u = u' = 0 \text{ at } x = 0,\ u' \to 1 \text{ as } x \to \infty. \qquad \ldots(2)$$

The special case $\beta = 0$, in which the wedge reduces to a flat plate, was numerically solved by Blasius[2]. But, the general case was discussed by Falkner and Skan[3]. The calculations were further extended by Hartree[4]. A rigorous proof of the solubility of the boundary value problems (1) and (2), for any non-negative constant β, was given by Weyl[5] .Weyl showed that the problem has a solution $u(x)$ whose first derivative increases with x and whose second derivative decreases and tends to zero as $x \to \infty$. Stewartson[6] tried to give further solutions of Falkner-Skan equation (1) in addition to those found by Hartree in order to understand the mechanism governing the theory of separation. Further Coppel[7] studied eqn. (1) from the stand-point of the

theory of equations and gave a direct and elementary proof of Weyl's theorem for all $\beta > 0$. Rosenhead[8] studied the equation of similar profiles

$$u''' + \alpha uu'' + \beta (1 - u'^2) + \gamma u'' = 0 \qquad ...(4)$$

(where $u - u(t)$ and primes denote derivatives w. r. t. the variable t) under the appropriate boundary condition

$$u = u' = 0 \text{ at } t = 0,\ u' \to 1 \text{ as } t \to \infty \qquad ...(5)$$

itogether with the side condition

$$0 < u' < 1 \text{ on } (0, \infty). \qquad ...(6)$$

Recent results, which did not take into account the side condition (6) were estabiseed by Hastings[9] and Troy[10]. But, a systematic and exhaustive bibliography of eqn. (4) was given by Gabutti[11] under the boundary condition (5). According to Gabutti, if $\alpha = -1$, $\beta > 0$, γ arbitrary, there is atleast one solution which satisfies $0 < u' < 1$ on $(0,\infty)$. By the additional condition $u' > 0$, or, alternatively $0 < \beta \leqslant 1$, the uniqueness of the solution was demonstrated. If $\alpha = 1$, $\beta < 0$, γ arbitrary, the existence of solutions for which $-1 < u' < 0$ in some initial interval $(0, t)$ and satisfying generally $u' > 1$ was established. In both the problems, Gabutti showed bounds for $u''(0)$ and qualitative behaviour of solutions. Firstly, the asymptotic behaviour of solutions of (1) was studied by Hartman[12].

In the same sequence, we have discussed the asymptotic behaviour of the profiles (4) under the same boundary and side conditions (5) and (6). We have shown interestng results in the solution of (4) in reference to the asymptotic behaviour of similar profiles.

Mathematical Analysis

Put $h(t) = 1 - u'(t)$ in the eqn. (4). Then, we get

$$h''(t) + (\alpha u + \gamma) h'(t) - \beta \{1 + u'(t)\} h(t) = 0. \qquad ...(7)$$

Taking $\alpha = 1$ and substituting

$$h(t) = x\, e^{-1/2 \int_0^t \{u(s) + \gamma\} ds}$$

eqn. (7) becomes

$$x'' - q(t)\, x = 0 \qquad ...(8)$$

where

$$q(t) = \tfrac{1}{2} u'(t) + \tfrac{1}{4} \{u(t) + \gamma\}^2 + \beta \{u'(t) + 1\}. \qquad ...(9)$$

From (9)

$$q'(t) = \tfrac{1}{2} \{u(t) + \gamma\} u'(t) + (\beta + \tfrac{1}{2}) u''(t)$$

and

$$q''(t) = \{\beta(\beta + \tfrac{1}{2}) + \tfrac{1}{4}\}\{u'(t)\}^2 + \tfrac{1}{2}\{u(t) + \gamma\}.\ u''(t) - (\beta + \tfrac{1}{2})\, u(t)\, u''(t) - \beta(\beta + \tfrac{1}{2}) - \gamma(\beta + \tfrac{1}{2})\, u''.$$

Since $0 < u' < 1$, $u'' > 0$, and $u' \sim 1$, $u \sim t$ as $t \to \infty$, hence for large t, $u \sim t \Rightarrow u + \gamma \sim t + \gamma\ (-\infty < \gamma < \infty)$.

Therefore, for large t, a constant C can be so chosen that

$$\frac{q'^2}{q^{5/2}} \leqslant C\left(\frac{u''^2}{(t+\gamma)^5} + \frac{1}{(t+\gamma)^3}\right)$$

and

$$\frac{|q''|}{q^{3/2}} \leqslant C\left(\frac{u''}{(t+\gamma)^2} + \frac{1}{(t+\gamma)^3}\right).$$

In addition, $\int^{\infty} u'\, dt$ is absolutely convergent (since $u'(t) \to 1$ as $t \to \infty$), so that

$$\int^{\infty} \frac{q'^2}{q^{5/2}}\, dt < \infty \text{ and } \int^{\infty} \frac{|q''|}{q^{3/2}}\, dt < \infty \qquad \ldots(10)$$

provided

$$\int^{\infty} \frac{u''^2}{(t+\gamma)^5}\, dt < \infty. \qquad \ldots(11)$$

Now the validity of (11) can be checked by integration by parts taking $\frac{u''}{(t+\gamma)^5}$ as the first function and u'' as the second function

$$\int \frac{u''}{(t+\gamma)^5}\, u''\, dt = \frac{u'\, u''}{(t+\gamma)^5} \int \frac{u'}{(t+\gamma)^5} \left\{ uu'' + \beta(1 - u'^2) + \gamma u'' - \frac{5u''}{t+\gamma} \right\} dt.$$

The above integral is absolutely convergent and $\liminf e''(t) = 0$

Therefore, (11) holds good.

Since $q(t)$ is a continuous and complex valued function satisfying the condition (10) as $t \to \infty$, the principal solution of differential equation (8) will be of the form

$$x \sim Cq^{-1/4}(t) \exp\left(-\int^{t} q^{1/4}(s)\, ds\right) \qquad \ldots(12)$$

where $C \neq 0$ is a constant, while the linearly independent solutions satisfy

$$x \sim C\, q^{-1/4}(t) \exp\left(\int^{t} q^{1/4}(s)\, ds\right). \qquad \ldots(13)$$

From (9)

$$q^{1/2}(t) = \frac{1}{2}\{u(t) + \gamma\} + \frac{1}{2}(2\beta + 1)\frac{u'(t)}{u(t)+\gamma} + \frac{\beta}{u(t)+\gamma} + 0\left(\frac{1}{t+\gamma}\right).$$

Also using $u + \gamma \sim t + \gamma$, we obtain from (9)

$$q^{1/4}(t) \sim \{\tfrac{1}{2}(t+\gamma)\}^{1/2}.$$

Therefore using $x \sim C\, q^{-1/4} \exp\left(-\int^{t} q^{1/2}(s)\, ds\right)$, we get

$$x \sim C\,(t+\gamma)^{-\beta-1} \exp\left(-\int^{t}\left[\frac{1}{2}\{u(s) + \gamma\} + \frac{\beta}{u(s)+\gamma}\right] ds\right). \qquad \ldots(14)$$

Also

$$x \sim C\, q^{-1/4}(t) \exp\left(\int^{t} q^{1/2}(s)\, ds\right)$$

gives

$$x \sim C\,(t+\gamma)^{\beta} \exp\left(\int^{t}\left[\frac{1}{2}\{u(s) + \gamma\} + \frac{\beta}{u(s)+\gamma}\right] ds\right). \qquad \ldots(15)$$

Using $h(t) \sim x \exp\left(-\frac{1}{2}\int_{0}^{t}\{u(s) + \gamma\}\, ds\right)$, the differential equation (7) has the principal solution satisfying

$$h \sim C\,(t+\gamma)^{-1-\beta} \exp -\left(\int^{t}\left[\left\{u(s) + \gamma\right\} + \frac{\beta}{u(s)+\gamma}\right] ds\right) \qquad \ldots(16)$$

while linearly independent solutions satisfy

$$h \sim C\,(t+\gamma)^{\beta} \exp\left(\int^{t} \frac{\beta\, ds}{u(s)+\gamma}\right), \quad C \neq 0. \qquad \ldots(17)$$

From eqn. (7), we have

$$h'''(t) + \{u(t) + \gamma\}\, h''(t) + (1 - 2\beta)\, u'(t)\, h'(t) = 0. \qquad \ldots(18)$$

Treating eqn. (18) as the second order equation in h' in the same way as (7) was handled, (18) has the principal solution satisfying

$$h' = C' (t + \gamma)^{-2\beta} \exp\left(- \int^{t} \{u(s) + \gamma\}\, ds\right) \qquad \ldots(19)$$

and linearly independent solutions satisfy

$$h' = C' (t + \gamma)^{-1+2\beta},\ C' \neq 0, \qquad \ldots(20)$$

$$\text{as } t \to \infty.$$

Theorem 1—If $\beta \geqslant 0$ and $u(t)$ be the solution of (4) — (6), then there exist constants $C_0 > 0$, C_1 such that (21) holds good as $t \to \infty$.

PROOF : For a given $u(t)$, we have to decide whether $h = 1 - u'(t)$ satisfies (16), (19) or (17), (20).

If $\beta \geqslant 0$, then (17) can hold good, for otherwise $h = 1 - u'(t) \to 0$, $t \to \infty$ is contradicted. But if $\beta \geqslant 0$, then the solution (16), (19); will hold good. But, we have deduced (21) from (16), (19), hence (21) will also hold good as $t \to \infty$ (since t has been taken sufficiently large, hence $t + \gamma$ will always be greater than zero i.e. positive).

Theorem 2—Let $\beta > 0$, $0 \leqslant \beta' < 1$, $\alpha \geqslant A(\beta, \beta')$, where $A(\beta, \beta')$, $\gamma(\alpha)$ are given in the theorem given below. Let $u(t)$ be the solution of (4) — (6). Then, there exist constants $C_0 > 0$, C_1 such that (21) holds good if and only if $u''(0) = \gamma'(\alpha')$; for other solutions $u(t)$ of (4) — (6), with

$$d' > A(\beta, \beta') \text{ and } 0 \leqslant u''(0) < \gamma'(\alpha')$$

the asymptotic relations (23) hold (with suitable constant $C_0 > 0$).

The special features of the symbols used in the Theorem 2 are clear from the following theorem :

*Theorem*3—Let β, β' be fixed, $\beta < 0$, and $0 \leqslant \beta' < 1$. Then there exists a number $A = A(\beta, \beta')$ and a continuous increasing function $\gamma'(\alpha')$ defined for $\alpha' \geqslant A$ with the properties that $\gamma'(A) = 0$ and that if $u(t)$ is a solution of (4) and $u(0) = \alpha'$, $u'(0) = \beta'$, then $u'(\infty) = 1$ and (6) holds if and only if $\alpha' \geqslant A$ and $0 \leqslant u''(0) \leqslant \gamma'(\alpha')$; in this case

$$u''(t) > 0 \text{ for } 0 < t < \infty.$$

The proof of Theorem 2 is based on the same pattern as used to prove Theorem 9.2 in Hartman[12] (p. 536).

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