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A CLASS OF EXACT SOLUTIONS OF STATIC CHARGED SPHERES IN GENERAL RELATIVITY

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A general method has been developed to solve the Einstein-Maxwell equations for the spherically symmetric charged bodies. Some particular solutions have also been discussed. It is hoped that many more exact solutions can be obtained by this method. It has also been observed that the pressure at the centre is negative and increases to zero at the surface for the spheres having charge only. The classical model of an electron has also been discussed.

1. INTRODUCTION

In the present paper we have obtained a class of solutions for the static charged fluid spheres. Many workers[1-9] have obtained different solutions for the same. Here we develop a general method to solve the field equations corresponding the static charged spheres having variable densities of matter and charge both. The solutions obtained by Mehra and Bohra[7] and Mehra[9] are the particular cases of this general method. Some new particular solutions have also been discussed. It is interesting to note that here we have obtained a solution corresponding to a sphere having only charge, no matter. In this solution, the pressure at the center is negative and increases to zero at the surface of the sphere. The known values of mass, radius and charge of the electron also satisfy the results obtained in this solution. It is hoped that many more particular solutions can be obtained by this method.

2. FIELD EQUATIONS

The line element

$$ds^2 = -e^{-\lambda(r)}\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + e^{\nu(r)}\,dt^2 \qquad \ldots(1)$$

is employed to solve the usual Einstein-Maxwell equations

$$G_\mu^\nu = -8\pi T_\mu^\nu \qquad \ldots(2)$$

for the static spherically symmetric charged spheres. For the case under study, the energy-momentum tensor T_μ^ν consists of two parts M_μ^ν and E_μ^ν for matter and charge respectively as

$$T^{\nu}_{\mu} = M^{\nu}_{\mu} + E^{\nu}_{\mu} . \qquad \ldots(3)$$

The non-zero components of M^{ν}_{μ} and E^{ν}_{μ} are

$$M^1_1 = M^2_2 = M^3_3 = -p(r), \quad M^4_4 = \rho(r) \qquad \ldots(4)$$

and

$$E^1_1 = -E^2_2 = -E^3_3 = E^4_4 = g^{11} g^{44} F^2_{14}/2 \qquad \ldots(5)$$

where F_{14} is given by[10]

$$\frac{d}{dr}[-g^{11} g^{44} F_{14} (-g)^{1/2}] = J^4 (-g)^{1/2}. \qquad \ldots(6)$$

Now we assume the charge density J^4 as an arbitrary function of r given by

$$J^4 = \psi(r) e^{-(\lambda+\nu)/2}. \qquad \ldots(7)$$

Putting the value of J^4 from eqn. (7) in eqn. (6), we get

$$F_{14} = [e^{(\lambda+\nu)/2} (\int \psi(r) r^2 dr)]/r^2 \qquad \ldots(8)$$

and leading to

$$E^1_1 = -E^2_2 = -E^3_3 = E^4_4 = (\int \psi(r) r^2 dr)^2/2r^4. \qquad \ldots(9)$$

Now the field equations (2) take the form

$$e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 = 8\pi[p - (\int \psi(r) r^2 dr)^2/2r^4] \qquad \ldots(10)$$

$$e^{-\lambda}[\nu''/r - \lambda'\nu'/4 + \nu'^2/4 + (\nu' - \lambda')/2r] = 8\pi[p + (\int \psi(r) r^2 dr)^2/2r^4] \qquad \ldots(11)$$

$$e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 = 8\pi[\rho + (\int \psi(r) r^2 dr)^2/2r^4]. \qquad \ldots(12)$$

In the above equations prime denotes differentiation with respect to r.

Any solution of physical significance of the above equations must satisfy the following conditions :

(i) It should match with Reissner-Nordstrom solution given by

$$F_{14} = \epsilon/r^2 \qquad \ldots(13)$$

$$e^{-\lambda} = e^{\nu} = 1 - 2M/r + 4\pi\epsilon^2/r^2 \qquad \ldots(14)$$

at the surface of the sphere i. e. $r = R$. Here M and ϵ are the constants of integration.

(ii) The pressure p must be positive inside the sphere and zero at the surface of the sphere.

(iii) The pressure p, F_{14}, e^{λ} and e^{ν} must be continuous throughout the sphere.

3. Solution of Field Equations

On eliminating p from eqn. (10) and (11) and substituting $e^{\nu} = y$ and $r^2 = x$, we have

$$e^{-\phi}\frac{d^2y}{dx^2} - \frac{1}{2}e^{-\phi}\bar{\phi}\frac{dy}{dx} + \frac{1}{4}\left[\frac{1-e^{-\phi}}{x^2} - \frac{e^{-\phi}\bar{\phi}}{x} - \alpha(x)\right]$$

$$y = 0 \qquad \ldots(15)$$

where

$$\phi(x) = [\lambda(r)]_{r^2=x} \qquad \ldots(16)$$

$$\alpha(x) = 8\pi[(\int \psi(r)\, r^2\, dr)^2/r^6]_{r^2=x} \qquad \ldots(17)$$

$$\bar{\phi}(x) = \frac{d\phi(x)}{dx}. \qquad \ldots(18)$$

To remove the first derivative of y from eqn. (15), we put

$$Z = \int e^{\phi/2}\, dx \qquad \ldots(19)$$

and we get

$$\frac{d^2y}{dz^2} + \frac{1}{4}\left[\frac{1-e^{-\phi}}{x^2} - \frac{\bar{\phi}\, e^{-\phi}}{x} - \alpha(x)\right]y = 0. \qquad \ldots(20)$$

On assuming

$$(1 - e^{-\phi})/x^2 - e^{-\phi}\bar{\phi}/x - \alpha(x) = 4S \text{ (constant)} \qquad \ldots(21)$$

eqn. (20) becomes

$$\frac{d^2y}{dz^2} + Sy = 0 \qquad \ldots(22)$$

having solution

$$Y^2 = e^{\nu} = \begin{cases} (C_1 \cos Z_1 + C_2 \sin Z_1)^2, & \text{for } S > 0 \\ (C_1 \cosh Z_1 + C_2 \sinh Z)^2, & S_1 < 0 \\ (C_1 Z + C_2)^2 & S = 0. \end{cases} \qquad \ldots(23)$$

Where C_1 and C_2 are constants of integration and

$$Z_1 = [S]^{1/2} Z. \qquad \ldots(24)$$

let us take

$$\psi(r) = \sum_{i=0}^{n} \psi_i r^i \qquad \ldots(25)$$

where $\psi_0, \psi_1, \psi_2, \ldots\ldots \psi_n$ are arbitrary constants.

Equation (8) with eqn. (25) gives

$$F_{14} = e^{(\lambda+\nu)/2} \sum_{i=0}^{n} \frac{\psi_i r^{i+1}}{i+3} . \qquad \ldots(26)$$

The constant of integration is taken to be zero to avoid the singularity at $r = 0$.

From eqn. (17) and (25) we get

$$\alpha(x) = 8\pi \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{\psi_i \psi_j x^{(i+j)/2}}{(i+3)(j+3)} . \qquad \ldots(27)$$

Again, we assume ρ as

$$\rho = \sum_{k=0}^{m} \rho_k r^k \qquad \ldots(28)$$

where $\rho_0, \rho_1, \rho_2, \ldots\ldots \rho_m$ are constants.

Equation (10) with eqns. (25) and (28) gives

$$e^{-\phi} = 1 - 8\pi \sum_{k=0}^{m} \frac{\rho_k x^{(k+2)/2}}{k+3} - 4\pi \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{\psi_i \psi_j}{(i+3)(j+3)} \frac{x^{(i+j+4)/2}}{(i+j+5)} .$$

$\ldots(29)$

The constant of integration is again taken zero to avoid the singularity at the centre. To satisfy eqn. (21), we must have

$$2S + 2\pi \sum_{k=0}^{m} \frac{k x^{(k-2)/2} \rho_k}{K+3} + \pi \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{(5i+5j+22) x^{(i+j)/2} \psi_i \psi_j}{(i+3)(j+3)(i+j+5)} = 0. \qquad \ldots(30)$$

Hence the coefficients of all different powers of x must be zero, so we must have

$$\rho_1 = 0$$

$$S = -\left(2\pi\rho_2 + 11\pi\psi_0^2/9\right)/5$$

$$\rho_l = -\frac{5l+12}{l}\sum_{q=0}^{l-2}\frac{\psi_q\,\psi_{l-q-2}}{(q+3)(l-q+1)},\quad l = 3, 4, \ldots 2n+2. \qquad \ldots(31)$$

We, therefore, conclude that $\psi_0, \psi_1, \ldots\ldots \psi_n$, ρ_0 and ρ_2 are arbitrary constants and S, $\rho_3, \rho_4, \ldots \rho_{2n+2}$ are constants which can be obtained by eqn. 31 and ρ_1 is always zero.

Case I When $S > 0$ – The pressure at any point r can be obtained from eqn, (10) as

$$p = \frac{S^{1/2}F_1(r)}{2\pi}\left[\frac{C_2\cos Z_1 - C_1\sin Z_1}{C_1\cos Z_1 + C_2\sin Z_1}\right] + F_2(r) \qquad \ldots(32)$$

where

$$F_1(r) = \left[1 - 8\pi\sum_{k=0}^{2n+2}\frac{\rho_k\,r^{k+2}}{k+3} - 4\pi\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{\psi_i\,\psi_j\,r^{i+j+4}}{(i+3)(j+3)(i+j+5)}\right]^{1/2} \qquad \ldots(33)$$

$$F_2(r) = \sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(i+j+4)\,r^{i+j+2}\,\psi_i\,\psi_j}{(i+3)(j+3)(i+j+5)} - \sum_{k=0}^{2n+2}\frac{\rho_k r^k}{k+3}. \qquad \ldots(34)$$

The zero presure and the continuity of F_{14}, e^{λ} and e^{ν} at $r = R$ gives

$$\epsilon = -\sum_{i=0}^{n}\frac{\psi_i\,R^{i+3}}{i+3} \qquad \ldots(35)$$

$$M = 4\pi\sum_{k=0}^{2n+2}\frac{\rho_k\,R^{K+3}}{k+3} + 2\pi\sum_{i=0}^{n}\sum_{j=0}^{n}\frac{(i+j+6)\,R^{i+j+5}\,\psi_i\,\psi_j}{(i+3)(j+3)(i+j+5)} \qquad \ldots(36)$$

$$C_1 = F_1(R)\cos Z_R + \frac{2\pi}{S^{1/2}}F_2(R)\sin Z_R \qquad \ldots(37)$$

$$C_2 = F_1(R)\sin Z_R - \frac{2\pi}{S^{1/2}}F_2(R)\cos Z_R \qquad \ldots(38)$$

where Z_R is the value of Z_1 when $x = R^2$. The mass of the sphere is given by eqn. (36). It is interesting to note that it has Newtonian gravitation mass and also the mass equivalence of electromagnetic energy of the sphere. So we conclude that the electromag-

netic energy of the sphere increases the mass of the body. This conclusion has also been drawn by Florides[9].

The pressure at the centre, p_c, is given by

$$p_c = \frac{S^{1/2}}{2\pi}\left[\frac{C_2 \cos Z_0 - C_1 \sin Z_0}{C_1 \cos Z_0 + C_2 \sin Z_0}\right] - \frac{\rho_0}{3} \qquad ...(39)$$

where ρ_0 and Z_0 are the values of density and Z_1 at $r = 0$ respectively. To get pressure positive and less than $\frac{1}{3}\rho_0$, at $r = 0$, we have

$$1 < \frac{3S^{1/2}}{2\pi\rho_0}\left[\frac{C_2 \cos Z_0 - C_1 \sin Z_0}{C_1 \cos Z_0 + C_2 \sin Z_0}\right] < 2. \qquad ...(40)$$

Case II : When $S < 0$—The calculation of this case is similar to the Calculation of case I and hence the results of this case can be easily obtained.

Case III : When $S = 0$—In this case e^ν and P are given by

$$e^\nu = (C_1 Z + C_2)^2 \qquad ...(41)$$

$$p = \frac{C_1}{2\pi (C_1 Z + C_2)} F_1(r) + F_2(r) \qquad ...(42)$$

where C_1 and C_2 are given by

$$C_1 = -2\pi F_2(R) \qquad ...(43)$$

$$C_2 = [e^{-\lambda/2} - C_1 Z]_{r=R}. \qquad ...(44)$$

To get Pressure positive and less than $\frac{\rho_0}{3}$, at $r = 0$, we have

$$1 < \frac{3C_1}{2\pi (C_1 Z + C_2)\rho_0} < 2. \qquad ...(45)$$

4. Particular Solutions

(A) Let $\psi = \psi_0$ (Constant). Then

$$\rho = \rho_0 + \rho_2 r^2 \qquad ...(46)$$

$$S = -\pi\left[2\rho_2 + (11/9)\psi_0^2\right]/5 \qquad ...(47)$$

where ρ_0, ρ_2 and ψ_0 are arbitrary constants. In this case possible solutions of physical significance are given below :

No.	ρ_0	ρ_1	S	Remarks
1	$\neq 0$	$\neq 0$	$\neq 0$	Already solved by Mehra[9]
2	$\neq 0$	$\neq 0$	$= 0$	Already solved by Mehra[9]
3	$\neq 0$	$= 0$	$\neq 0$	Already solved by Mehra and Bohra[7]
4	$= 0$	$= 0$	$\neq 0$	To be discussed here
5	$= 0$	$= 0$	$= 0$	Flat space

Case IV—In this case charged sphere has only charge, no matter inside it. Penney[10] has also obtained this type of solution for the classical model of an electron. He used the field equations

$$G_{\mu\nu} - g_{\mu\nu}/4 = -KT_{\mu\nu} \qquad ...(48)$$

in place of eqns. (2). The results in our case are given below;

$$F_{14} = [\psi_0 r\, e^{(\lambda+\nu)/2}]/3 \qquad ...(49)$$

$$e^{-\lambda} = 1 - 4\pi\, \psi_0^2\, r^4/45 \qquad ...(50)$$

$$e^{\nu} = (C_1 \cosh\, Z_1 + C_2 \sinh\, Z_1)^2 \qquad ...(51)$$

$$p = \frac{\psi_0}{2}\left[\frac{11}{45\pi}\left(1 - \frac{4\pi\psi_0^2\, r^4}{45}\right)\right]^{1/2} \frac{C_1 \sinh Z_1 + C_2 \cosh Z_1}{C_1 \cosh Z_1 + C_2 \sinh Z_1} + \frac{2}{45}\, \psi_0^2\, r^2 \qquad ...(52)$$

where

$$Z_1 = (11)^{1/2} \sin^{-1} [2\, (\pi/5)^{1/2}\, \psi_0\, r^2/3]. \qquad ...(53)$$

The values of constants of integration are given by

$$\epsilon = \psi_0\, R^3/3, \qquad ...(54)$$

$$M = 4\pi\, \psi_0^2\, R^5/15, \qquad ...(55)$$

$$C_1 = (1 - 4\pi\, \psi_0^2\, R^4/45)^{1/2} \cosh Z_R + [4\, (\pi/55)^{1/2} \sinh Z_R]\, /3 \qquad ...(56)$$

$$C_2 = (1 - 4\pi\, \psi_0^2\, R^4/45)^{1/2} \cosh Z_R - [4\, (\pi/55)^{1/2} \sinh Z_R]/3, \qquad ...(57)$$

where Z_R is the value of Z_1 at $r = R$. The pressure at the centre is given by

$$p_c = C_2/\, C_1. \qquad ...(58)$$

In this case $C_2/C_1 < 0$, so the pressure is negative at the centre and increases to zero at the surface of the sphere. It shows that the nature of charge is different from the nature of matter where the pressure is positive at the centre and decreases to zero at the surface of the sphere. Perhaps due to this difficulty Penney has modified the usual field equation (2) to equation (48) to discuss the classical model of the electron.

In relativistic units the mass and the radius of the electron are 7×10^{-56} and 2×10^{-13} respectively and the value of $4\pi\epsilon^2$ is 2.1×10^{66}. These values are also satisfied by equations (54) and (55). The values of ψ_0 from eqn. (54) or (55) for electron is

given as $\approx 1.52 \times 10^4$ which is density of charge inside the electron. The pressure at the centre is $\approx -6.7 \times 10^2$ while $e^{-\lambda}$ and e^{ν} equal to ≈ 1. We therefore, can assume the electron as a charged ball having only charge of density $1.52 \times 1\,0^4$ inside it.

5. Discussion

The general method to obatin exact solutions of Einstein-Mexwell equations has been given here. We have also discussed some particular solutions. To obtain a new particular solution, first we have to assume ψ and then $e^{-\phi}$ must be calculated. The values of arbitrary constants ρ_0 and ρ_2 may be taken in such a way so that $e^{\phi/2}$ becomes integrable to give Z. After finding Z, solution can easily be obtained as we have obtained here. The arbitrary constants $\psi_0, \psi_1, \ldots \psi_n$ are to be chosen in such a way which gives positive density and pressure at the centre of the body.

The above x solutions also satisfy the necessary regularity conditions at centre namely

$$\frac{\partial p}{\partial r} = \frac{\partial \nu}{\partial r} = \frac{\partial \lambda}{\partial r} = 0.$$

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