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ON APPLICATIONS OF LATTICE FIXED POINT THEOREM

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In the present paper a lattice fixed point theorem involving two operators is proved using lattice fixed point theorem due to Tarski[3] and nonlinear contraction principle of Boyd and Wong[2] and then apply our results to nonlinear integral equation of mixed type.

INTRODUCTION

The fixed point theorems are used as handy tools in proving the existence and uniqueness of the solutions of certain differential and integral equations. In this paper we establish the existence of extremal solutions of certain nonlinear mixed type of integral equation with the application of the lattice fixed point theorem due to Tarski[3]. First we prove the fixed point theorem involving two operators and then we apply our results to nonlinear integral equations of mixed type. In the following we give some definitions which will be used in the subsequent discussion of the paper.

Definition 1 – An operator A on a Banach space X with a norm $\| \cdot \|$ is said to be contraction if for $x, y \in X$

$$\| Ax - Ay \| \leqslant a \| x - y \| \qquad \ldots(1.1)$$

for some $0 \leqslant a < 1$.

Definition 2 – An operator A on a Banach space X is said to be nonlinear contraction if for $x, y \in X$,

$$\| Ax - Ay \| \leqslant \phi (\| x - y \|) \qquad \ldots(1.2)$$

where ϕ is a continuous real function satisfying $\phi (r) < r$, for $r > 0$.

Remark 1 : It is very easy to see that every contraction is a nonlinear contraction.

Below we give two fixed point theorems which we will employ in our discussion.

Theorem A [ref. 2]—Let $T : X - X$ is a nonlinear contraction on a Banach space X then T has unique fixed point in X.

Theorem B [ref. 3]—Let L be a nonempty set with order relation $\leqslant$, with

(i) $(L, \leqslant)$ is complete lattice,

(ii) $T : L \to L$ be isotone,

(iii) $F = \{u : T(u) = u, u \in L\}$.

Then the set F is nonempty and $(F, \leqslant)$ is a complete lattice. We give our main results in the following section.

2. Main Results

Theorem 1 – Let S be a closed and bounded subset of a Banach Algebra X and A and B be two operators on S satisfying the following assumptions :

(H_1) $(S, \leqslant)$ is a complete lattice,

(H_2) A is a nonlinear contraction and B is a bounded with bound M, on S,

(H_3) $(\frac{I}{A})^{-1} B$ is isotone increasing on S,

(I denotes the identity operator on S)

(H_4) $Ax\ By \in S$ for every $x, y, \in S$.

Then equation

$$Ax\ Bx = X \qquad \ldots(2.1)$$

has a solution in S and set of all solutions is a complete lattice.

Before proving Theorem 1 below we give some remarks.

Remark 2 : The hypothesis (H_2) of Theorem 1 guarantees the existence of

$$(\frac{I}{A})^{-1}.$$

Remark 3 : We note that if A and B are isotone increasing then $\left(\frac{I}{A}\right)^{-1} B$ is isotone increasing.

Proof of Theorem 1—Equation (2.1) can be written as

$$Tx = x, x \in S, \qquad \ldots(2.2)$$

where $T = \left(\frac{I}{A}\right)^{-1} B$. The operator T is by (H_3), isotone increasing on S. Let $y \in S$ and define A_y on S by

$$A_y(x) = Ax\ By,\ X \in S. \qquad \ldots(2.3)$$

Now by (H_4) A_y maps S into itself.

Further $X_1, X_2 \in S$ one has

$$\|A_y(x_1) - A_y(x_2)\| = \| A(x_1)\ By \| - A(x_2)\ By\|$$

$$= \|By\|\ \|A(x_1) - A(x_2)\|. \qquad \ldots(2.4$$

Since A is nonlinear contraction, we may choose the constant a such that $a < 1$ and

$$\| A(x_1) - A(x_2)\| \leqslant \frac{a}{M}\|x_1 - x_2\|$$

$$M\ \|A_y(x_1) - A_y(x_2)\| \leqslant a\ \|x_1 - x_2\|,\ a < 1.$$

Hence the operator A_y is a contraction, and therefore there exists a unique $Y' \in S$ such that

$$A_y(y') = y'. \qquad \ldots(2.5)$$

By the definition of T which shows that $Ty = y'$. Then T maps S into itself. Now assertion follows directly by the application of the Theorem B. This completes the proof.

Let us consider the mixed type of nonlinear integral equation.

$$x(t) = h(t) \int_0^a k\ (t, r)\ g\ (r, x\ (r))\ dr$$

$$+ \left(\int_0^t v\ (t, s)\ f\ (s, x\ (s))\ ds\right)\left(\int_0^a k\ (t, r)\ g\ (r, x\ (r))\ (dr)\right) \qquad \ldots(2.6)$$

where $0 \leqslant s \leqslant t \leqslant a$, and where all the functions involved on right-hand side of (2.6) are real-valued. Equation (2.6) is a general nonlinear mixed type of integral equation in the sense that Voltera and Hammerstein integral equations are particular cases of it, which can be shown as follows :

1. Let

$$\int_0^a k(t, r)\, g(r, x(r))\, dr = 1, 0 \leqslant t \leqslant a.$$

Then integral equation (2.6) reduces to nonlinear Voltera integral equation namely,

$$x(t) = h(t) + \int_0^t v(t, s) f(s, x(s))\, ds, \; 0 \leqslant t \leqslant a. \qquad \ldots(2.7)$$

2. Let

$h(t) = 0, 0 \leqslant t \leqslant a$ and

$$\int_0^t v(t, s) f(s, x(s))\, ds = 1, \; 0 \leqslant t \leqslant a.$$

Then integral equation (2.6) reduces to nonlinear Hammerstein integral equation,

$$x(t) = \int_0^a k(t, v)\, g(r, x(r))\, dr, \; 0 \leqslant t \leqslant a. \qquad \ldots(2.8)$$

Many such type of nonlinear integral equations are studied in Krasnoselskii[4] with the help of fixed point theorems for existence and the uniqueness of the solution. In this paper we shall obtain existence of minimal and maximal solution of nonlinear integral equation (2.6). This will be established by the application of our Theorem 1, under suitable conditions.

Let X be the space of real bounded functions on $I = [0, a]$ which are also Lebesgue integrable. For any $x \in X$

$$\| X \| = \sup_{t \in I} | x(t) | . \qquad \ldots(2.9)$$

Thus X is a bounded Banach Algebra with this supremum norm.

Let $x, y \in X$, then by $x \leqslant y$ we mean $x(t) \leqslant y(t)$ $t \in I$, then $(X, \leqslant)$ is a Dedekind complete lattice.[1]

Now we suppose that the following assumptions hold.

(A_1) $f(s, x), g(s, x)$ are real functions on $I \times R$ which are monotone nondecreasing in x for each $s \in I$ and $f(s, x)$ satisfies a Lipschitz condition with respect to x with a Lipschitz constant L, i.e. for any $x_1, x_2 \in R$, there is a constant $L > 0$, such that

$$| f(s, x_1) - f(s, x_2) | \leqslant L | x_1 - x_2 | .$$

(A_2) The functions $f(s, \phi(s))$ and $g(s, \phi(s))$ are Lebesgue integrable for every Lebesgue integrable function $\phi(s)$ on I.

(A_3) The functions $h(t)$, $f(s, x)$, $g(s, x)$, $v(t, s)$ and $k(t, s)$ are bounded for $s, t \in I$ and $| x | < \infty$, with bounds, H_0, K_1, K_2, V_0, and K_0 respectively.

(A_4) The functions $v(t, s)$ and $k(t, s)$ are nonnegative.

Theorem 2— Let the assumptions (A_1) to (A_4) hold, then then the integral equation (2.6) posseses minimal and maximal solutions on I whenever $LV_0 a < 1$.

PROOF : Define a subset S of the Banach space X by

$$S = \{ x \in X \mid \| x \| \leqslant M \}. \qquad \ldots(2.10)$$

where $M = (H_0 + V_0 K_1 a)(K_0 K_2 a)$ whenever $a \geqslant 1$,

and $M = (H_0 + V_0 K_1)(K_0 K_2 a)$ for $a < 1$.

Clearly S is a closed and bounded subset of X. Since $(x, \leqslant)$ is a Dedekind complete lattice[1], $(s, \leqslant)$ is also complete lattice. Let us define two operators A and B on S by

$$(Ax)(t) = h(t) + \int_0^t v(t, s) f(s, x(s))\, ds, \; t \in I, \qquad \ldots(2.11)$$

and

$$(By)(t) = \int_0^a k(t, r)\, g(r, y(r))\, dr, \; t \in I. \qquad \ldots(2.12)$$

We define the operator T on S by

$$Tx = Ax\, Bx, \; x \in S. \qquad \ldots(2.13)$$

Now using the hypothesis (A_3) we see that $Ax\ By \in S$ whenever $x, y \in S$. In particular $Ax\ Bx \in S$ i.e. $Tx \in S$ for every $x \in S$ which implies that T maps S into itself. The monotone nondecreasing property of $f(s, x)$, $g(s, y)$ and the nonnegative nature of $v(t,s)$ and $k(t, s)$ imply that A and B are isotone increasing, and therefore the operator T is a isotone increasing on S. Now we show that A is a strict contraction. For, let $x_1, x_2 \in S$, using (A_1) we obtain

$$\| (A\ x_1)(t) - (A\ x_2)(t) \| \leqslant u \| x_1 - x_2 \| \qquad \ldots(2.14)$$

where $u = L\ V_0\ a < 1$. This shows that A is a contraction on S. Also the operator B is bounded on S. Now all the conditions of Theorem 1 are satisfied by T. Hence by an application of it leads to the facts that the operator equation $Tx = x$ has a solution and set of all solutions is a complete lattice. This further implies that eqn. (2.6) has a unique minimal and maximal solution on I. This completes the proof.

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REFERENCES

1. G. Birkhoff, *Lattice Theory*. Amer. Soc. Coll. Publ. Vol. 25, New York. 1967, p. 361.
2. D. Boyd, and J.S.W. Wong, *Proc. Amr. Math. Soc*, **20** (1969), 458-46.
3. A. Tarski, *Pacific J. Math.* 5 (1955), 285-310.
4. M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*. Pergamon (student Editions) New York, 1963.