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FIXED POINT THEOREMS IN METRIC, 2-METRIC AND NORMED LINEAR SPACES

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In this paper we show the existence of a common fixed point for four self-maps on a metric space under certain contraction and asymptotic regularity type conditions and obtain an analogous result in 2-metric spaces. We also prove that under certain conditions the set of all common fixed points of four self-maps on a closed convex subset of a strictly convex normed linear space is closed and convex. We also observe that certain theorems of Khan and Swaleh[1] are false and suggest some suitable modifications for them.

Khan and Swaleh[1] obtained two fixed point theorems for there self-maps, one of them being in metric space set up and the other being in normed linear space set up. Unfortunately both the theorems are found to be false. The work in this paper is an outcome of the authors' attempt to modify them.

Throughout this paper : R^+ is the set of all non-negative real numbers; and $\phi : R^+ \to R^+$ is an increasing map with $\phi(t+) < t \; \forall \; t \in (0,\infty)$ and

$$\lim_{t \to +\infty} (t - \phi(t)) > 0.$$

We begin with the following :

Lemma – Let $\{\alpha_{n,m}\}$ and $\{\beta_{n,m}\}$ be double sequences of non-negative real valued functions on a non-empty set X. Suppose that

$$\alpha_{n,m}(x) - \phi(\alpha_{n,m}(x)) \leqslant \beta_{n,m}(x)$$

$$(x \in X; \; n, m = 1, 2.....)$$

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and

$\beta_{n,m} \to 0$ uniformly on X as both n and $m \to +\infty$.

Then $\alpha_{n,m} \to 0$ uniformly on X as both n and $m \to +\infty$.

PROOF : Suppose $\alpha_{n,m} \not\to 0$ uniformly on X as both n and $m \to +\infty$. Then $\exists\, \epsilon > 0$, strictly increasing sequences $\{x_{mk}\}$, $\{n_k\}$ of positive integers and a sequence $\{x_k\}$ in X such that

$$r_k \geqslant \epsilon \ (k = 1, 2, \ldots\ldots) \qquad \ldots(\mathrm{I})$$

where $r_k = \alpha_{n_k, m_k}(x_k)$.

Let $s_k = \beta_{n_k, m_k}(x_k) \quad (k = 1,2,\ldots)$.

From the hypothesis it is clear that $\{s_k\}$ converges to zero and

$$r_k - \phi(r_k) \leqslant s_k \ (k = 1,2,\ldots\ldots). \qquad \ldots(\mathrm{II})$$

If $\{r_k\}$ is unbounded then it admits a subsequence diverging to $+\infty$ so that on using the fact that $\{s_k\}$ converges to zero one can conclude from inequality (II) that $\varliminf_{t \to +\infty} (t - \phi(t)) \leqslant 0$ which is a contradiction to the hypothesis.

Hence $\{r_k\}$ is bounded.

Since a bounded sequence of real numbers admits a convergent subsequence, the seq uence $\{r_k\}$ admits a convergent subsequence with limit, say, l. Taking limits on both sides of inequality (II) as r_k varies over the terms of the subsequence we obtain $l \leqslant \phi(l+)$. Since $\phi(t+) < t \ \forall\ t \in (0, \infty)$, we must have $l = 0$. This is a contradiction in view of inequality (I). Hence the lemma.

We shall now obtain a fixed point theorem for four self-maps on a metric space.

Theorem 1— Let f, g, S and T be self-maps on a complete metric space (X, d). Suppose that

$$\begin{aligned} d(fx, gy) \leqslant \phi\ (\max\{&d(Sx, Ty) + d(Sx, fx) + d(Ty, gy), \\ &d(Sx, fx) + d(Sx, gy), \\ &d(Ty, gy) + d(fx, Ty)\}) \\ &+ k_1 d(Sx, fx) + k_2 d(Ty, gy) \end{aligned} \qquad \ldots(\mathrm{III})$$

for all x, y in X, where k_1, k_2 are non-negative real constants. Further suppose that there are sequences $\{x_n\}$, $\{y_n\}$ in X such that the sequences $\{d(fx_n, Sx_n)\}$ and $\{d(gy_n, Ty_n)\}$ converge to zero. Then the sequences $\{fx_n\}$, $\{Sx_n\}$, $\{gy_n\}$ and $\{Ty_n\}$ all converge to to the same limit z for some $z \in X$. Further the following statements hold :

(i) If S is continuous at z and f, S commute weakly then $Sz = z$. If, further, $k_1 = 0$ then $fz = z$.

(ii) If T is continuous at z and g, T commute weakly then $Tz = z$. If, further, $k_2 = 0$ then $gz = z$.

(iii) Neither the pair f, S nor the pair g, T can have a common fixed point other than z.

PROOF : From triangle inequality we have

$$d(Sx_n, Ty_m) \leqslant d(Sx_n, fx_n) + d(fx_n, gy_m) + d(gy_m, Ty_m),$$

$$d(Sx_n, gy_m) \leqslant d(Sx_n, fx_n) + d(fx_n, gy_m)$$

and

$$d(fx_n, Ty_m) \leqslant d(fx_n, gy_m) + d(gy_m, Ty_m).$$

Taking $x = x_n$ and $y = y_m$ in inequality (III) and then making use of the above three inequalities one can show that

$$a_{n,m} \leqslant \phi(a_{n,m} + 2b_n + 2c_m) + k_1 b_n + k_2 c_m \quad (n, m = 1, 2, \ldots),$$

where

$$a_{n,m} = d(fx_n, gy_m), \; b_n = d(Sx_n, fx_n)$$

and

$$c_m = d(Ty_m, gy_m).$$

Hence

$$a_{n,m} + 2b_n + 2c_m - \phi(a_{n,m} + 2b_n + 2c_m) \leqslant (2 + k_1) b_n + (2 + k_2) c_m \quad (n, m = 1, 2, \ldots).$$

Since the sequences $\{b_n\}$, $\{c_m\}$ converge to zero, from the Lemma it follows that $a_{n,m} + 2b_n + 2c_m \to 0$ as both n and $m \to +\infty$ and hence $a_{n,m} \to 0$ as both n and $m \to +\infty$. We have $d(fx_n, fx_m) \leqslant a_{n,m} + a_{m,m} \to 0$ as both n and $m \to +\infty$. Hence $\{fx_n\}$ is Cauchy.

We have $d(gy_n, gy_m) \leqslant a_{n,m} + a_{n,n} \to 0$ as both n and $m \to +\infty$. Hence $\{gy_n\}$ is Cauchy.

Since (X, d) is complete, the sequences $\{fx_n\}$ and $\{gy_n\}$ are convergent.

Since $a_{n,n} \to 0$ as $n \to +\infty$, the sequences $\{fx_n\}$ and $\{gy_n\}$ have the same limit z for some $z \in X$. Since the sequences $\{b_n\}$ and $\{c_n\}$ converge to zero it is clear that the sequences $\{Sx_n\}$ and $\{Ty_n\}$ also converge to z.

(i) Suppse S is continuous at z and f, S commute weakly. Since S is continuous at z, $\{Sfx_n\}$ converges to Sz. We have $d(fSx_n, Sfx_n) \leqslant d(Sx_n, fx_n) \to 0$ an $n \to \infty$.

Hence $\{fSx_n\}$ converges to Sz.

Taking $x = Sx_n$ and $y = y_n$ in inequality (III) and then taking limits on both sides as $n \to \infty$ we obtained

$$d(Sz, z) \leqslant \phi\ (d(Sz, z) +).$$

Since $\phi(t+) < t\ \forall\ t \in (0, \infty)$ we must have $Sz = z$.

Suppose $k_1 = 0$. Now taking $x = z$ and $y = y_n$ in inequality (III) and then taking limits on both sides as $n \to \infty$ and making use of the fact that $Sz = z$ we obtain

$$d(fz, z) \leqslant \phi\ (d(z, fz) +) \text{ so that } fz = z.$$

The proof of (ii) is analogous to that of (i).

(iii) Suppose f and S have a common fixed point w. Taking $x = w$ and $y = y_n$ in inequality (III) and then taking limits on both sides as $n \to \infty$, we obtain

$$d(w, z) \leqslant \phi\ (d(w, z) +) \text{ so that } w = z.$$

Hence the mappings f and S can have no common fixed point other than z.

In a similar manner it can be shown that the mappings g and T also cannot admit a common fixed point other than z.

Corollary 1—Let f, g, S and T be self-maps on a complete metric space (X, d). Suppose that

$$d(fx, gy) \leqslant \alpha_1\ (d(Sx, Ty) + \alpha_2\ d(Sx, fx) + \alpha_3\ d(Ty, gy) + \alpha_4\ d(fx, Ty) + \alpha_5\ d(Sx, gy) \qquad \ldots(IV)$$

for all x, y in X, where $\alpha_1, \alpha_2, \ldots, \alpha_5$ are non-negative real constants such that

$$\max\{\alpha_1 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_4, \alpha_3 + \alpha_5\} < 1. \qquad \ldots(V)$$

Further suppose that S, T are continuous on X; (f, S), (g, T) are weakly commuting pairs and that there are sequences $\{x_n\}$, $\{y_n\}$ in X such that the sequences $\{d(fx_n, Sx_n)\}$ and $\{d(gy_n, Ty_n)\}$ converge to zero. Then f, g, S and T have a unique common fixed point z which is also the unique common fixed point of f and S and also of g and T.

PROOF : Let $\alpha = \max\{\alpha_1 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_4, \alpha_3 + \alpha_5\}$. From inequality (IV), for all x, y in X, we have

$$\begin{aligned} d(fx, gy) &\leqslant \alpha_1 d(Sx, Ty) + \alpha_2 d(Sx, fx) + \alpha_3 d(Ty, gy) \\ &\quad + \alpha_4 (d(fx, Sx) + d(Sx, Ty)) \\ &\quad + \alpha_5 (d(Sx, Ty) + d(Ty, gy)) \\ &= (\alpha_1 + \alpha_4 + \alpha_5) d(Sx, Ty) + (\alpha_2 + \alpha_4) d(Sx, fx) \\ &\quad + (\alpha_3 + \alpha_5) d(Ty, gy) \\ &\leqslant \alpha (d(Sx, Ty) + d(Sx, fx) + d(Ty, gy)). \end{aligned}$$

Hence inequality (III) holds for all x, y in X with

$$\phi(t) = \alpha t \ (t \in R^+) \text{ and } k_1 = k_2 = 0.$$

Since $0 \leqslant \alpha < 1$ it is clear that ϕ is increasing on R^+, $\phi(t+) < t \ \forall \ t > 0$ and $\lim_{t \to +\infty} (t - \phi(t)) = +\infty$.

Hence the Corollary.

Remark 1 : Example 1 shows that in Corollary 1 inequality (V) cannot be replaced by

$$\max\{\alpha_1 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_5\} < 1$$

even though X is compact; S, T are identity maps on X and $x_n = y_n \ \forall \ n = 1, 2, \ldots$. In particular, the example shows that Theorem 3.1 of Khan and Swaleh[1] is false. Corollary 1 with $S = T$ and $x_n = y_n$ is a possible modification of it.

Example 1—Let $X = [0, 1]$ with the usual metric.

Define self-maps f, g on X as

$$fx = \begin{cases} \frac{x}{4} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

and

$$gx = \frac{x}{4}.$$

Then

$$|fx - gy| \leqslant |fx - x| + \tfrac{1}{3}|y - gy| \ \forall\ x, y \in X.$$

For $x_n = y_n = \frac{1}{n}$ $(n = 1, 2, \ldots)$ the sequences $\{|fx_n - x_n|\}$ and $\{|gy_n - y_n|\}$ converge to zero.

But f has no fixed point.

We shall now state without proof an analogue of Theorem 1 in 2-metric spaces. Before doing so we introduce the concept of weak commutativity of two self-maps on a 2-metric space and provide an example.

Definition—Let f, g be self-maps on a 2-metric space (X, d). We say that f, g commute weakly on X or f, g are weakly commutative on X or (f, g) is a weakly commuting pair on X if

$$d(fgx, gfx, a) \leqslant d(gx, fx, a) \ \forall\ x \in X \text{ and } \forall\ a \in X.$$

Remark 2 : A commuting pair of self-maps on a 2-metric space is weakly commutative but the converse need not be true as Example 2 illustrates it.

Example 2— Let $X = \{1, 2, 3, 4\}$.

Define $d : X \times X \times X \to R^+$ as

$$d(x, y, z) = \begin{cases} 0 \text{ if } x = y \text{ or } y = z \text{ or } z = x \{x, y, z\} = \{1, 2, 3\} \\ \frac{1}{2} \text{ if } x, y, z \text{ are distinct and } \{x, y, z\} \neq \{1, 2, 3\}. \end{cases}$$

Define self-maps f, g on X as $f1 = f2 = 3$, $f3 = f4 = 2$ and $g1 = g3 = 2$, $g2 = g4 = 3$.

Clearly d is a 2-metric on X and f, g are weakly commutative on X. But f, g do not commute.

Theorem 2—Let f, g, S and T be self-maps on a complete 2-metric space (X, d). Suppose that

$$d(fx, gy, a) \leqslant \phi(\max\{d(Sx, Ty, a) + d(Sx, fx, a) + d(Ty, gy, a), d(Sx, fx, a) + d(Sx, gy, a)\})$$

(equation continued on p. 608)

$$d(Ty, gy, a) + d(fx, Ty, a)\})$$
$$+ k_1 d(Sx, fx, a) + k_2 d(Ty, gy, a)$$

for all x, y, a in X, where k_1, k_2 are non-negative real constants. Further suppose that there are sequences $\{x_n\}$, $\{y_n\}$ in X such that the sequences $\{d(fx_n, Sx_n, a)\}$ and $\{d(gy_n, Ty_n, a)\}$ converge to zero uniformly on X. Then the sequences $\{fx_n\}$, $\{Sx_n\}$, $\{gy_n\}$ and $\{Ty_n\}$ all converge to the same limit z for some $z \in X$. Further neither the pair f, S nor the pair g, T can have a common fixed point other than z. Moreover, statements (i) and (ii) of Theorem 1 hold here also, provided d is continuous on X.

We shall now obtain a fixed point theorem in normed linear spaces.

Theorem 3—Let X be a strictly convex normed linear space, K be a closed convex subset of X, and f, g, S and T be self-maps on K. Suppose that S and T are continuous affine maps on K and

$$\|fx - gy\| \leqslant \max \{\|Sx - Ty\|, \tfrac{1}{2}(\|Sx - fx\| + \|Ty - gy\|),$$
$$\alpha \|fx - Ty\| + (1 - \alpha)\|Sx - gy\|,$$
$$\beta \max \{\|fx - Ty\|, Sx - gy\|\} + (1 - \beta) \max \{\|Sx - Ty\|,$$
$$\tfrac{1}{2}(\|Sx - fx\| + \|Ty - gy\|)\},$$
$$\Psi(\max \{\|fx - Ty\|, \|Sx - gy\|\})\} \qquad \ldots(VI)$$

for all x, y in K, where α, β are constants in $[0, 1)$ with $\alpha > 0$ and $\Psi : R^+ \to R^+$ is an increasing map with $\Psi(t+) < t \; \forall \; t \in (0, \infty)$. Then the set of all common fixed points of f, g, S and T is closed and convex.

PROOF : Let $F = \{x \in K / fx = gx = Sx = Tx = x\}$. Let $\{z_n\}$ be a sequence in F converging to some $z \in K$. Since S is continuous on K we have $Sz = \lim_{n \to \infty} Sz_n = \lim_{n \to \infty} z_n = z$. Similarly from the continuity of T we have $Tz = z$.

Taking $x = z$ and $y = z_n$ in inequality (VI) and then taking limits on both sides as $n \to \infty$ and making use of the fact that $Sz = z$ we obtain

$$\|fz - z\| \leqslant \max \{\tfrac{1}{2}\|fz - z\|, \alpha \|fz - z\|, \tfrac{1}{2}(1 + \beta)\|fz - z\|,$$
$$\Psi(\|fz - z\| +)\}.$$

Since $0 < \alpha < 1$, $0 \leqslant \beta < 1$ and $\Psi(t+) < t \; \forall \; t \in (0, \infty)$, we must have $fz = z$.

In a similar manner it can be shown that $gz = z$.

Hence $z \in F$.

Hence F is closed.

Let $w_1, w_2 \in F$ and $w = \frac{1}{2}(w_1 + w_2)$.

Since S is affine we have $Sw = \frac{1}{2} Sw_1 + \frac{1}{2} Sw_2 = \frac{1}{2}(w_1 + w_2) = w$.

Since T is also affine we must have $Tw = w$.

Taking $x = w$ and $y = w_1$ in inequality (VI) we obtain

$$\|fw - w_1\| \leqslant \max \{\|w - w_1\|, \tfrac{1}{2}\|w - fw\|,$$

$$\alpha \|fw - w_1\| + (1 - \alpha)\|w - w_1\|,$$

$$\beta \max \{\|fw - w_1\|, \|w - w_1\|\} + (1 - \beta)$$

$$\max \{\|w - w_1\|, \tfrac{1}{2}\|w - fw\|\},$$

$$\Psi (\max \{\|fw - w_1\|\}, \|w - w_1\|\})\}.$$

Since $\|w - w_1\| = \frac{1}{2}\|w_1 - w_2\|$, $\|w - fw\| \leqslant \|w - w_1\| + \|w_1 - fw\|$, $0 < \alpha < 1$, $0 \leqslant \beta < 1$ and $\Psi(t+) < t \; \forall \; t \in (0, \infty)$, from the above inequality it is clear that

$$\|fw - w_1\| \leqslant \tfrac{1}{2}\|w_1 - w_2\|. \qquad \ldots(\text{VII})$$

In a similar manner it can be shown that

$$\|fw - w_2\| \leqslant \tfrac{1}{2}\|w_1 - w_2\|. \qquad \ldots(\text{VIII})$$

From triangle inequality and inequalities (VII) and (VIII) it is clear that

$$\|w_1 - fw\| + \|fw - w_2\| = \|w_1 - w_2\|.$$

Since X is a strictly convex normed linear space it now follows that fw lies on the line segment joining w_1 and w_2. Now from inequalities (VII) and (VIII) it is evident that $fw = \frac{1}{2}(w_1 + w_2) = w$. In a similar manner it can be shown that $gw = w$.

Hence $w \in F$.

Since F is closed and the mid-point of any two points of F lies in F, the set F must be convex.

Corollary 2—Let X be a strictly convex normed linear space, K be a closed convex subset of X, and f, g, S and T be self-maps on K. Suppose that S and T are continuous affine maps and

$$\|fx - gy\| \leqslant \alpha_1 \|Sx - Ty\| + \alpha_2 \|Sx - fx\| + \alpha_3 \|Ty - gy\| + \alpha_4 \|fx - Ty\| + \alpha_5 \|Sx - gy\| \quad \ldots(IX)$$

for all x, y in K, where $\alpha_1, \alpha_2, \ldots, \alpha_5$ are non-negative real constants with

$$\max \{\alpha_4, \alpha_5\} < 1 \quad \ldots(X)$$

and

$$\max \{\alpha_1 + 2\alpha_2 + \alpha_4 + \alpha_5, \alpha_1 + 2\alpha_3 + \alpha_4 + \alpha_5\} \leqslant 1. \quad \ldots(XI)$$

Then the set of all common fixed points of f, g, S and T is closed and convex.

PROOF : We may assume that at least one of $\alpha_1, \alpha_2, \ldots, \alpha_5$ is positive.

Case (i) $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

If at least one of α_4, α_5 is zero then it is clear that inequality (IX) implies inequality (VI) with $\Psi(t) = (\max \{\alpha_4, \alpha_5\})\, t$. Suppose now that both α_4 and α_5 are positive.

Let

$$\alpha = \frac{\alpha_4}{\alpha_4 + \alpha_5}.$$

Then $0 < \alpha < 1$.

From inequality (IX) we have

$$\|fx - gy\| \leqslant (\alpha_4 + \alpha_5)(\alpha \|fx - Ty\| + (1 - \alpha) \|Sx - gy\|)$$

$$\leqslant \alpha \|fx - Ty\| + (1 - \alpha) \|Sx - gy\|$$

so that inequality (VI) holds in this case also.

Case (ii) $\max \{\alpha_1, \alpha_2, \alpha_3\} > 0$.

Let

$$\beta = \frac{\alpha_4 + \alpha_5}{\alpha_1 + \max \{2\alpha_2, 2\alpha_3\} + \alpha_4 + \alpha_5}.$$

Then $0 \leqslant \beta < 1$.

From inequality (IX) we have

$$\begin{aligned}
\|fx - gy\| &\leqslant \alpha_1 \|Sx - Ty\| + \max\{\alpha_2, \alpha_3\}(\|Sx - fx\| + \|Ty - gy\|) \\
&\quad + (\alpha_4 + \alpha_5)\max\{\|fx - Ty\|, \|Sx - gy\|\} \\
&\leqslant (\alpha_1 + \max\{2\alpha_2, 2\alpha_3\})\max\{\|Sx - Ty\|, \\
&\quad \tfrac{1}{2}(\|Sx - fx\| + \|Ty - gy\|)\} \\
&\quad + (\alpha_4 + \alpha_5)\max\{\|fx - Ty\|, \|Sx - gy\|\} \\
&\leqslant (\alpha_1 + \max\{2\alpha_2, 2\alpha_3\} + \alpha_4 + \alpha_5) \\
&\quad [\beta \max\{\|fx - Ty\|, \|Sx - gy\|\} \\
&\quad + (1 - \beta)\max\{\|Sx - Ty\|, \tfrac{1}{2}(\|Sx - fx\| + \|Ty - gy\|)\}] \\
&\leqslant \beta \max\{\|fx - Ty\|, \|Sx - gy\|\} + (1 - \beta)\max\{\|Sx - Ty\|, \\
&\quad \tfrac{1}{2}(\|Sx - fx\| + \|Ty - gy\|)\}.
\end{aligned}$$

Hence inequality (VI) holds.

Thus inequality (IX) always implies inequality (VI) when α, β and Ψ are suitably selected.

Hence the Corollary.

Remark 3 : Condition (IX) is vacuously satisfied when $\alpha_4 \geqslant 1$ and $g = T$ or when $\alpha_5 \geqslant 1$ and $f = S$.

Remark 4 : Example 3 shows that in Corollary 2 one cannot conclude either the closedness or the convexity of the set F of all common fixed points of f, g, S and T if condition (X) is dropped, even though K is compact, and f, S and T are identity maps on K. Example 4 shows that in Corollary 2 one cannot conclude the convexity of F if the term α_1 is dropped from the left hand side of inequality (XI), even though K is compact; S, T are identity maps on K; $f = g$ and f is continuous. In particular, Example 3 shows that Theorem 3.3 of Khan and Swaleh[1] is false and Example 4 shows that the set F need not be convex under the hypothesis of Theorem 3.3 of Khan and Swaleh[1]. Corollary 2 with $S = T$ is a possible modification of Theorem 3.3 of Khan and Swaleh[1].

Example 3—Let $X = R$, the set of all real numbers regarded as a normed linear space.

Let $K = [0, 1]$.

Define

$$g : K \to K \text{ as } gy = \begin{cases} y \text{ if } y \neq \frac{1}{2} \\ 0 \text{ if } y = \frac{1}{2} \end{cases}.$$

Then $F = \{y \in K / gy = y\} = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ is neither closed nor convex.

Clearly inequality (IX) is satisfied with $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\alpha_5 = 1$, provided f, S and T are taken as identity maps on K.

Example 4—Let X and K be as in Example 3.

Define

$$f : K \to K \text{ as } fx = x^2.$$

Then

$$| fx - fy | \leqslant 2 | x - y | \ \forall \ x, y \in K.$$

But $F = \{x \in K / fx = x\} = \{0, 1\}$ is not convex.

Clearly inequality (IX) is satisfied with $\alpha_1 = 2$, $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$, provided S, T are taken as identity maps on K and $g = f$.

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