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THE g-INVERSE AND REVERSE-CONVERSE SERIES INVERSE

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Let $\left\{ A_k^n \right\}$ and $\left\{ B_k^n \right\}$ be a pair of mixed orthogonal number arrays, i. e. $\Sigma_k \; A_k^n \, B_j^k = \Sigma_k \; A_n^k B_k^j = \delta_j^n$. Let g be astrictly increasing number-theoretic function with $g(k) \geqslant 0$ and $g^{-1}(k) \geqslant 0$. We establish the g-inverse that

$$F(n) = \sum_{0 \leqslant k \leqslant g^{-1}(n)} A_{g(k)}^n \, f(k) \qquad \ldots(1)$$

implies

$$f(n) = \sum_{0 \leqslant k \leqslant g(n)} B_k^{g(n)} \, F(k). \qquad \ldots(2)$$

The usual direct converse that (2) implies (1) is false in the general. However, we may reverse the role of g and establish what we call a reverse-converse g-inverse that

$$f(n) = \sum_{0 \leqslant k \leqslant g^{-1}(n)} B_{g(k)}^n \, F(k) \qquad \ldots(3)$$

implies

$$F(n) = \sum_{0 \leqslant k < g(n)} A_k^{g(n)} f(k). \qquad \ldots(4)$$

The usual converse of this is also false in general. When g is the identity function, then (1) and (4) are the same, as also (2) and (3), and in this case our theorems become the ordinary 'if and only if' standard inversion theorem known throughout the mathematical literature. Our general theorems do not seem to have been noticed in the literature before, and what special instances we have found there have been proved on the spot without any cognizance that the phenomena are completely general. It therefore seems worthwhile to set down these simple results together with a number of interesting app-

lications to special functions, combinatorics, and number theory. Rotated forms of our theorems (which follow at once by looking at matrix transposes) are given and we raise unsolved problems concerning the existence of mixed orthogonal relations that would give us 'if and only if' theorems.

1. Introduction

It is well-known, and easy to show, that

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) \qquad \ldots(1.1)$$

if and only if

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} F(k). \qquad \ldots(1.2)$$

It is not so well-known that for all natural numbers s

$$F(n) = \sum_{k=0}^{[n/s]} (-1)^{sk} \binom{n}{sk} f(k) \qquad \ldots(1.3)$$

implies

$$f(n) = \sum_{k=0}^{sn} (-1)^k \binom{sn}{k} F(k). \qquad \ldots(1.4)$$

The usual converse that (1.4) implies (1.3) is false in general, as almost any values of n and s will provide counterexamples.

There is, however, what we shall call a reverse-converse which says that

$$f(n) = \sum_{k=0}^{[n/s]} (-1)^{sk} \binom{n}{sk} F(k) \qquad \ldots(1.5)$$

implies

$$F(n) = \sum_{k=0}^{sn} (-1)^k \binom{sn}{k} f(k). \qquad \ldots(1.6)$$

This case seems pretty trivial, but we shall see below that the reverse-converse is not always as simple to see.

Earlier[2] we gave a new series inversion involving binomial coefficients which is of the 'if and only if' form (although we explicitly stated and proved the implication

in one direction only; our later papers bringing out the two-way nature of the implications in more general settings)

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+bk}{n} f(k) \qquad \ldots(1.7)$$

if and only if

$$\binom{a+bn}{n} f(n) = \sum_{k=0}^{n} (-1)^k \frac{a+bk-k}{a+bn-k} \binom{a+bn-k}{n-k} F(k). \qquad \ldots(1.8)$$

Riordan[11] (pp. 50 – 51) rewrites this inversion in the form

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{p+qk-k}{n-k} b_k \qquad \ldots(1.9)$$

if and only if

$$b_n = \sum_{k=0}^{n} \frac{p+qk-k}{p+qn-k} \binom{p+qn-k}{n-k} a_k \qquad \ldots(1.10)$$

which follows from (1.7) – (1.8) by setting $a = p$, $b = q$ and $b_k = \binom{p+qk}{k} f(k)$.

The motivation for our paper is that Singhal and Kumari[12] have recently offered a one-way generalized inversion which says that for $s = 1, 2, \ldots$

$$a_n = \sum_{k=0}^{[n/s]} (-1)^{n-sk} \binom{p+qsk-sk}{n-sk} b_k \qquad \ldots(1.11)$$

implies

$$b_n = \sum_{k=0}^{sn} \frac{p+qk-k}{p+qsn-k} \binom{p+qsn-k}{sn-k} a_k. \qquad \ldots(1.12)$$

Their proof is a step-by-step parallel of Riordan's proof of our inversion pair (1.9) – (1.10), but Singhal-Kumari say nothing about whether the converse holds or not.

Again, it is left to the reader to choose some simple values of n, s, p, q to find easy counter-examples to the standard converse.

Singhal and Kumari evidently were motivated to find that (1.11) implies (1.12) in order to invert the polynomial

$$g_n^c(x, r, s) = \sum_{k=0}^{[n/s]} \frac{(c + rk)_{n-sk}}{(n - sk)!} \gamma_k x^k \qquad \ldots(1.13)$$

so as to express x^n as a linear combination of the g_k^c polynomials. Rather than dwell on any special cases we shall move forward to the new theorems.

2. The g-inverse

We assume as a standing hypothesis that we have a pair $\left\{A_k^n\right\}, \left\{B_k^n\right\}$ of mixed orthogonal number arrays, i. e. in the terminology introduced in Gould[9] that

$$\sum_{k=j}^{n} B_k^n \; A_j^k = \sum_{k=j}^{n} B_j^k \; A_k^n = \delta_j^n \; . \qquad \ldots(2.1)$$

'Covariant' orthogonality

$$\sum_{k=j}^{n} B_k^n \; A_k^j = \delta_j^n \qquad \ldots(2.2)$$

and 'contravariant' orthogonality

$$\sum_{k=j}^{n} B_n^k \; A_j^k = \delta_j^n \qquad \ldots(2.3)$$

will not concern us here. Many of the common number arrays, e. g. binomial coefficients, Stirling numbers of first and second kind, etc. enjoy the mixed type orthogonality.

Theorem 1 – Let g be an increasing number-theoretic function with $g(k) \geqslant 0$ and $g^{-1}(k) \geqslant 0$. Then we have the g-inverse that

$$F(n) = \sum_{0 \leqslant k < g^{-1}(n)} A_{g(k)}^n \, f(k) \qquad \ldots(2.4)$$

implies

$$f(n) = \sum_{0<k<g(n)} B_k^{g(n)} F(k). \qquad \ldots(2.5)$$

PROOF : First we note that $j \leqslant g^{-1}(k)$ implies $g(j) \leqslant k$ which will allow us to interchange summation ranges easily. Also we need to remark that $g(n) = g(j)$ only for $n = j$ so that $\delta_{g(j)}^{g(n)} = \delta_j^n$. The remaining details of the proof mimic the case where g is the identity function. We have in fact

$$\sum_{0<k<g(n)} B_k^{g(n)} F(k) = \sum_{0<k<g(n)} B_k^{g(n)} \sum_{0<j<g^{-1}(k)} A_{g(j)}^k f(j)$$

$$= \sum_{0<j<n} f(j) \sum_{g(j)<k<g(n)} B_k^{g(n)} A_{g(j)}^k$$

$$= \sum_{0<j<n} f(j)\, \delta_j^n = f(n). \text{Q.E.D.}$$

To see why the usual converse is false, consider that we should have

$$\sum_{0<k<g^{-1}(n)} A_{g(k)}^n f(k) = \sum_{0<k<g^{-1}(n)} A_{g(k)}^n \sum_{0<j<g(k)} B_j^{g(k)} F(j)$$

$$= \sum_{0<j<n} F(j) \sum_{g^{-1}(j)<k<g^{-1}(n)} A_{g(k)}^n B_j^{g(k)}$$

but we do not know anything about

$$\sum_{g^{-1}(j)<k<g^{-1}(n)} A_{g(k)}^n B_j^{g(k)} \qquad \ldots(2.6)$$

since not every subscript on A nor every superscript on B appears so this expression is hardly expected to be a Kronecker delta. Comparison of (2.6) with legitimate Kronecker delta

$$\sum_{g(j)<k<g(n)} B_k^{g(n)} A_{g(j)}^k \delta_j^n \qquad \ldots(2.7)$$

makes it clear why the ordinary converse cannot be expected to hold.

3. THE REVERSE-CONVERSE g-INVERSE

We reverse the role of g in our work and obtain

Theorem 2—Let g be an increasing number-theoretic function write $g(k) \geqslant 0$ and $g^{-1}(k) \geq 0$. Then we have the reverse-converse g-inverse that

$$f(n) = \sum_{0 \leqslant k \leqslant g^{-1}(n)} B^n_{g(k)} F(k) \qquad \ldots(3.1)$$

implies

$$F(n) = \sum_{0 \leqslant k < g(n)} A^{g(n)}_k f(k). \qquad \ldots(3.2)$$

PROOF : We have

$$\sum_{0 \leqslant k \leqslant g(n)} A^{g(n)}_k f(k) = \sum_{0 < k \leqslant g(n)} \sum_{0 \leqslant j \leqslant g^{-1}(k)} B^k_{g(j)} F(j)$$

$$= \sum_{0 \leqslant j \leqslant n} F(j) \sum_{g(j) < k < g(n)} A^{g(n)}_k B^k_{g(j)}$$

$$= \sum_{0 < j < n} F(j) \delta^n_j = F(n). \text{ Q.E.D.}$$

Again the usual converse is not possible to prove since we know nothing in advance about the expression

$$\sum_{g^{-1}(j) \leqslant k \leqslant g^{-1}(n)} B^n_{g(k)} A^{g(k)}_j . \qquad \ldots(3.3)$$

The success of Theorem 2, of course, depends upon the legitimate Kronecker delta

$$\sum_{g(j) \leqslant k < g(n)} A^{g(n)}_k B^k_{g(j)} = \delta^n_j . \qquad \ldots(3.4)$$

Before we delineate applications, some other variations need to be mentioned, which is the purpose of the next section.

4. *p*-SHIFT AND ROTATED INVERSE FORMS

As noted by Riordan[11] since $\delta^{n+p}_{j+p} = \delta^n_j$ for any integer p, then it is trivial to see that Theorem 1 remains true when we replace the coefficients by $A^{n+p}_{g(k)+p}$ and $B^{g(n)+p}_{k+p}$ repectively. The same is true for Theorem 2 and these forms will be called "p-shifted" inverse pairs.

Thus the pair (1.5) – (1.6) leads to the p-shifted pair

$$F(n) = \sum_{k=0}^{[n/s]} (-1)^{sk} \binom{n+p}{sk+p} f(k) \qquad \ldots(4.1)$$

implies

$$f(n) = \sum_{k=0}^{sn} (-1)^k \binom{sn+p}{k+p} F(k) \qquad \ldots(4.2)$$

for any natural number s and any integer p.

Another kind of inverse pair noted by Riordan in dubbed by him a "rotated" pair. This arises by simply looking at matrix transposes. Thus, in Riordan's terminology, (1.1) and (1.2) in rotated form says that

$$F(n) = \sum_{k=n}^{\infty} (-1)^k \binom{k}{n} f(k) \qquad \ldots(4.3)$$

if and only if

$$f(n) = \sum_{k=n}^{\infty} (-1)^k \binom{k}{n} F(k). \qquad \ldots(4.4)$$

Of course, both (1.1), (1.2) and (4.3), (4.4) follow from the classical and elementary mixed orthogonality for binomial coefficients

$$\sum_{k=j}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{j} = \delta_j^n \qquad \ldots(4.5)$$

which is evident from the binomial theorem. It is easy to see that a rotated of (1.3) and (1.4) is that

$$F(n) = \sum_{n/s \leqslant k \leqslant \infty} (-1)^{sk} \binom{sk}{n} f(k) \qquad \ldots(4.6)$$

implies

$$f(n) = \sum_{sn \leqslant k \leqslant \infty} (-1)^k \binom{k}{sn} F(k). \qquad \ldots(4.7)$$

Our general rotation theorem is as follows.

Theorem 3—With g as before, we have the rotated g-inverse that

$$F(n) = \sum_{g^{-1}(n) \leqslant k \leqslant \infty} A_n^{g(k)} f(k) \qquad \ldots(4.8)$$

implies

$$f(n) = \sum_{g(n) \leqslant k \leqslant \infty} B_{g(n)}^k F(k). \qquad \ldots(4.9)$$

PROOF : We have

$$\sum_{g(n) \leqslant k \leqslant \infty} B_{g(n)}^k F(k) = \sum_{g(n) \leqslant k \leqslant \infty} B_{g(n)}^k \sum_{g^{-1}(k) \leqslant j \leqslant \infty} A_k^{g(j)} f(j)$$

$$= \sum_{n \leqslant j \leqslant \infty} f(j) \sum_{g(n) \leqslant k \leqslant g(j)} B_{g(n)}^k A_k^{g(j)}$$

$$= \sum_{n \leqslant j \leqslant \infty} f(j) \delta_n^j = f(n). \text{ Q.E.D.}$$

Again the usual direct converse is generally false. But we can prove the expected reverse-converse form, which is

Theorem 4—With g as before, we have the reverse-converse rotated g-inverse that

$$f(n) = \sum_{g^{-1}(n) \leqslant k \leqslant \infty} B_n^{g(k)} F(k) \qquad \ldots(4.10)$$

implies

$$F(n) = \sum_{g(n) \leqslant k \leqslant \infty} A_{g(n)}^k f(k). \qquad \ldots(4.11)$$

PROOF : We have

$$\sum_{g(n) \leqslant k \leqslant \infty} A_{g(n)}^k f(k) = \sum_{g(n) \leqslant k \leqslant \infty} A_{g(n)}^k \sum_{g^{-1}(k) \leqslant j \leqslant \infty} B_k^{g(j)} F(j)$$

$$= \sum_{n \leqslant j \leqslant \infty} F(j) \sum_{g(n) \leqslant k \leqslant g(j)} A_{g(n)}^k B_k^{g(j)}$$

$$= \sum_{n \leqslant j \leqslant \infty} F(j) \delta_n^j = F(n). \text{ Q.E.D.}$$

These four theorems, along with the p-shift technique, allow us to set down easily number of remarkable inverse pairs. This rich variety seems to have gone unnoticed before. Thus nothing like these results is to be found in Riordan[11] or other expected source listings in the literature. We shall look briefly at the Singhal-Kumari inverse and others like it.

5. Rothe and Abel Inverse Applications

Riordan[11] pays particular note to four classes of inverse relations: Legendre, Chebyshev, Abel and Gould. From some points of view they are all tied together into two broad categories : Rothe and Abel. The distinction, we mean is between Rothe coefficients $A_k(a, b) = \frac{a}{a + bk}\binom{a + bk}{k}$ and Abel coefficients $B_k(a, b) = \frac{a}{a + bk}(a + bk)^k/k!$.

We have already set down in (1.11) – (1.12) the g-inverse that arises from (1.9) and (1.10)when $g(k) = s\,k$, $A_k^n\ (= 1)^{n-k}\binom{p + qk - k}{n - k}$, and $B_k^n = \frac{p + qs - k}{p + qn - k}\binom{p + qn - k}{n - k}$. The reverse-converse Singhal-Savita Kumari inverse pair follows from Theorem 2, and says that for s a natural number,

$$b_n = \sum_{k=0}^{[n/s]} \frac{p + qsk - sk}{p + qn - sk}\binom{p + qn - sk}{n - sk} a_k \qquad \ldots(5.1)$$

implies

$$a_n = \sum_{k=0}^{sn} (-1)^{sn-k}\binom{p + qk - k}{sn - k} b_k. \qquad \ldots(5.2)$$

The usual converse of this, of course, is generally false.

Note that the Singhal-Savita Kumari and its reverse-converse are but one single instance where we choose $g(k) = sk$, s being a natural number. We could just as easily take any other allowable g and our theorems immediately give us correct inverse pairs, no further work being necessary. Thus one can take $g(k) = k^2$, with $g^{-1}(k) = \sqrt{k}$ and write down an astonishing-looking set of formulas! Perhaps we need to exhibit such a pair to emphasize the richness of our theorems. Thus, this choice of g gives us from (1.1) and (1.2) that

$$F(n) = \sum_{k=0}^{[\sqrt{n}]} (-1)^{k^2}\binom{n}{k^2} f(k) \qquad \ldots(5.3)$$

implies

$$f(n) = \sum_{k=0}^{n^2} (-1)^k \binom{n^2}{k} F(k). \qquad \ldots(5.4)$$

The transform discussed in (Gould[4], p. 326)

$$G(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{a+n+bk}{n} F(k)$$

and its inverse

$$\binom{a+n+bn}{n} f(n) = \sum_{k=0}^{n} (-1)^k \frac{a+k+bk+1}{a+n+bn+1} \binom{a+n+bn+1}{n-k} G(k)$$

is another practice example where we can find any number of g-inverse extensions and reverse-converses. These will have applications like the Singhal-Kumari pair to numerous generalized polynomial systems.

For a simple illustration of the Abel case, recall[11] (p.99) that a very simple inverse pair is

$$F(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x(x+n-k)^{n-k-1} f(k)$$

if and only if

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x(x-n+k)^{n-k-1} F(k).$$

Choose the function $g(k) = sk$ again. We have at once by Theorem 1 that

$$F(n) = \sum_{k=0}^{[n/s]} (-1)^{sk} \binom{n}{sk} x(x+n-sk)^{n-sk-1} f(k) \qquad \ldots(5.5)$$

implies

$$f(n) = \sum_{k=0}^{sn} (-1)^k \binom{sn}{k} x(x-sn+k)^{sn-k-1} F(k). \qquad \ldots(5.6)$$

The possible choices are certainly unlimited.

For another example of the usefulness of our theorems, we shall set down the results when we choose $g(k) = sk$ and use the inversion proved by Gould and Hsu[8]. Let $\{a_i\}$ and $\{b_i\}$ be two sequences of numbers such that

$$\psi(x, n) = \prod_{i=1}^{n} (a_i + b_i x) \neq 0 \qquad \ldots(5.7)$$

for all nonnegative integers x, n, with $\psi(x, 0) = 1$. Then for any natural number s we have that

$$F(n) = \sum_{k=0}^{sn} (-1)^{sk} \binom{n}{sk} \psi(sk, n) f(k) \qquad \ldots(5.8)$$

implies

$$f(n) = \sum_{k=0}^{[n/s]} (-1)^k \binom{sn}{k} (a_{k+1} + kb_{k+1}) \psi(sn, k+1)^{-1} F(k). \qquad \ldots(5.9)$$

The reverse-converse is just as easily determined by Theorem 2. The rotated form would be just as easily written down and would generalize the rotation given Gould and Hsu[8] (p. 887).

Finally we got down the reverse-converse g-inverse of (1.7) and (1.9) $g(k) = sk$. We have at once by Theorem 2 that for any natural number s

$$\binom{a + bn}{n} f(n) = \sum_{k=0}^{[n/s]} \frac{a + bsk - sk}{a + bn - sk} \binom{a + bn - sk}{n - sk} F(k) \qquad \ldots(5.10)$$

implies

$$F(n) = \sum_{k=0}^{sn} (-1)^{sn-k} \binom{sn}{k} \binom{a + bk}{sn} f(k). \qquad \ldots(5.11)$$

6. Application to Stirling Numbers

We may illustrate our results by making an application to the familiar Stirling numbers of first and second kinds in Riordan's popular notation. Recall that these are defined, respectively, by the expansions

$$\binom{x}{n} n! = \sum_{k=0}^{n} s(n, k) x^k \text{ and } x^n = \sum_{k=0}^{n} S(n, k) \binom{x}{k} k! \qquad \ldots(6.1)$$

and they enjoy the mixed othogonality conditions

$$\sum_{k=j}^{n} S(n,k)\, s(k,j) = \sum_{k=j}^{n} s(n,k)\, S(k,j) = \delta_j^n . \qquad \ldots(6.2)$$

It is therefore not surprising that we may write down at once the following inverse pa irs using the linear function $g(k) = sk$, s being a natural number.

$$F(n) = \sum_{k=0}^{[n/s]} S(n, sk)\, f(k) \text{ implies } f(n) = \sum_{k=0}^{sn} s(sn, k)\, F(k); \qquad \ldots(6.3)$$

$$f(n) = \sum_{k=0}^{[n/s]} s(n, sk)\, F(k) \text{ implies } F(n) = \sum_{k=0}^{sn} S(sn, k)\, f(k); \qquad \ldots(6.4)$$

$$F(n) = \sum_{k=0}^{sn} S(n,[k/s])\, f(k) \text{ implies } f(n) = \sum_{k=0}^{[n/s]} s([n/s], k)\, F(k) \qquad \ldots(6.5)$$

$$f(n) = \sum_{k=0}^{sn} s(n, [k/s])\, F(k) \text{ implies } F(n) = \sum_{k=0}^{[n/s]} S([n/s], k)\, f(k); \qquad \ldots(6.6)$$

$$F(n) = \sum_{k=0}^{[n/s]} S(n+p, sk+p)\, f(k) \text{ implies } f(n) = \sum_{k=0}^{sn} s(sn+p, k+p)\, F(k); \qquad \ldots(6.7)$$

and the reader may write down rotated forms of all these.

7. Combined gh-Inverses and Reverse-Converses

By introducing two functions, g and h, we may reasonably expect to be able to combine a g-inverse and an h-inverse (which is a reverse-converse of the g-inverse) into a single pair of formulae which yield upon suitable specialization either the g-inverse or the reverse-converse. To illustrate this we take up formulae (4.1) and (4.2) and combine them into a single pair.

Consider the two distinct formulas

$$F(n) = \sum_{k=0}^{[rn/s]} (-1)^{sk} \binom{rn+p}{sk+p} f(k) \qquad \ldots(7.1)$$

and

$$f(n) = \sum_{k=0}^{[sn/r]} (-1)^{rk} \binom{sn+p}{rk+p} F(k) \qquad \ldots(7.2)$$

where r and s are natural numbers. Clearly :

$$\begin{cases} \text{If } r = 1 \text{ than (7.1) implies (7.2);} \\ \text{If } s = 1 \text{ then (7.2) implies (7.1);} \\ \text{If } r = s = 1 \text{ then (7.1) if and only if (7.2).} \end{cases} \quad \ldots(7.3)$$

In the same manner, we may combine the Singhal-Kumari inverse pair (1.11) and (1.12) with the corresponding reverse-converse pair (5.1) and (5.2) and get the the following small

Theorem 5 – Let r and s be natural numbers. Define

$$F(n) = \sum_{k=0}^{[rn/s]} (-1)^{sk} \binom{p + qsk - sk}{rn - sk} f(k) \quad \ldots(7.4)$$

and

$$f(n) = \sum_{k=0}^{[sn/r]} (-1)^{rk} \frac{p + qrk - rk}{p + qsn - rk} \binom{p + qsn - rk}{sn - rk} F(k). \quad \ldots(7.5)$$

Then the following implications hold : (i) if $r = 1$ (7.4) implies (7.5); (ii) if $s = 1$ (7.5) implies (7.4); (iii) if $r = s = 1$ then (7.4) if and only if (7.5).

The general situation is elegant and brings out the symmetry or duality of our theorems. We have obviously

Theorem 6 – Let g and h be increasing number-theoretic functions with $g(k) \geqslant 0$, $g^{-1}(k) \geqslant 0$, $h(k) \geqslant 0$, and $h^{-1}(k) \geqslant 0$. Let A and B satisfy mixed orthogonality relations (2.1). Define

$$F(n) = \sum_{0 \leqslant k \leqslant g^{-1}h(n)} A_{g(k)}^{h(n)} f(k) \quad \ldots(7.6)$$

and

$$f(n) = \sum_{0 \leqslant k \leqslant h^{-1}g(n)} B_{h(k)}^{g(n)} F(k). \quad \ldots(7.7)$$

Recall that $j \leqslant g^{-1}(k)$ implies $g(j) \leqslant k$ and $j \leqslant h^{-1}(k)$ implies $h(j) \leqslant k$ and that $\delta_{g(j)}^{g(n)} = \delta_{h(j)}^{h(n)} = \delta_j^n$. Then the following implications hold true : (i) if h is the identity function, then (7.6) implies (7.7); (ii) if g is the identity function, then implies (7.7); if g is the identity function, then (7.7) implies (7.6); (iii) if g and h are both the

identity function, then (7.6) holds if and only if (7.7) holds (this being the classical case).

8. Applications in Number Theory

The ideas we have used so far have an immediate realization in number theory in that we may at once write down g-inverse generalized Möbius pairs. Here is a simple example.

Theorem 7—Let s be a natural number and suppose that $s \mid n$. Then

$$F(n) = \sum_{d \mid \frac{n}{s}} f(d) \qquad \ldots(8.1)$$

implies

$$f(n) = \sum_{d \mid sn} \mu\left(\frac{sn}{d}\right) F(d). \qquad \ldots(8.2)$$

Remark : The usual converse is again not generally true. The case $s = 1$ in the classical Möbius inversion formula and is then an 'if and only if'. Of course this g-inverse has the corresponding reverse converse given by

Theorem 8—Let r be a natural number and suppose that $r \mid n$. Then

$$f(n) = \sum_{d \mid \frac{n}{r}} \mu\left(\frac{n}{rd}\right) F(d) \qquad \ldots(8.3)$$

implies

$$F(n) = \sum_{d \mid rn} f(d). \qquad \ldots(8.4)$$

Combining together we get

Theorem 9—Let $r \mid n$ and $s \mid n$. Define

$$F(n) = \sum_{d \mid \frac{rn}{s}} f(d) \qquad \ldots(8.5)$$

and

$$f(n) = \sum_{d \mid \frac{sn}{r}} \mu\left(\frac{sn}{rd}\right) F(d). \qquad \ldots(8.6)$$

Then the following implications hold : (i) if $r = 1$ then (8.5) implies (8.6); (ii) if $s = 1$ then (8.6) implies (8.5); (iii) if $r = s = 1$ then (8.5) holds if and only if (8.6) holds, this being the classical Möbius inversion pair.

9. Conclusion

On examining the theorems we have presented we find that everything follows from some pretty trivial consequences of g being assumed to be a strictly increasing, non-negative, number-theoretic function. Yet the theorems do not seem to reside visibly in the past literature, and so we feel they are worth this study. We cannot resist setting down two more examples :

$$F(n) = \sum_{k=0}^{[n/s]} \binom{x}{n - sk} \text{ implies } f(n) = \sum_{k=0}^{sn} \binom{-x}{sn - k} F\{k\} \qquad \ldots(9.1)$$

and

$$F(n) = \sum_{k=0}^{sn} (-1)^{[k/s]} \binom{n}{[k/s]} f(k) \text{ implies } f(n) = \sum_{k=0}^{[n/s]} (-1)^k \binom{[n/s]}{k} F(k).$$

The first is an extension of a formula cited by Riordan[11] (p. 43) and the second is a variation on (1.1) – (1.4) by choosing $g(k) = [k/s]$.

We mention several unsolved problems. For what kinds of mixed orthogonal arrays A, B, are (2.6) and/or (3.3) Kronecker deltas? Are there other kinds of g-extensions of the classical case ($s = 1$) that yield 'if and only if' inversions?

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