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PROJECTIVE STRUCTURES IN SPACE-TIME

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We introduce the concept of projective mapping in space-time and show that, while on one hand these mappings have the property of retaining the geometric nature of curves and surfaces of space-time in contrast to the other better known mappings, e.g., harmonic or conformal, and thus preserve the hierarchy of causality, on the other they can be used as a distance decreasing device (in some sense) between two globally hyperbolic (Lorentzian) manifolds, provided their Ricci curvatures are suitably restricted.

We observe that projective parameter p (an essential ingredient of projective maps) leads to a 5-dimensional manifold which embeds the given manifold M and which is similar to the manifold proposed by Kaluza-Klein as a means to unification of field theories. From this point of view geodesics of M become the trajectories of particles with zero charge. In the case of manifolds with parallel Ricci-tensor, parameter p also helps to alter incompleteness of time-like geodesics into completeness.

INTRODUCTION

In this paper we consider those geometric objects on space-time which are invariants of projective maps (to be defined shortly). Classically two symmetric affine connections, Γ, $\bar{\Gamma}$ on a differentiable manifold are said to be projectively related if for a suitable covector field $\{p_i\}$:

$$\bar{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk} + \delta^{i}_{j}\ p_k + \delta^{i}_{k}\ p_j \qquad \ldots(0.1)$$

is true.

It is known[3,4,25,26] that the system of geodesics corresponding to Γ and $\bar{\Gamma}$ is same, although their arc-length parameters differ, and hence geodesics are projectively invariant but not their affine parameters.

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In contrast, a parameter p related to the affine parameter s through the differential equation*

$$\frac{p'''}{p'} - \frac{3}{2}\left(\frac{p''}{p'}\right)^2 = \frac{2}{n-1} R_{jk} \frac{dx^j}{ds}\frac{dx^k}{ds} \qquad \ldots(0.2)$$

to the Ricci curvature on n-dimensional manifold is, however, known to be projective invariant[11,27] (up to linear fractional transformations $p \to \frac{ap+b}{cp+d} ad - bc \neq 0$ for real numbers a, b, c, d).

Another object which is independent of projective change (0.1) is the so called projective connection[21]:

$$\Pi^l_{jk} = \Gamma^l_{jk} - \frac{1}{n+1}\left(\delta^l_j \Gamma^a_{ak} + \delta^j_k \Gamma^a_{aj}\right). \qquad \ldots(0.3)$$

It is known[21] that differential equation :

$$\frac{d^2x^i}{dp} + \Pi^i_{jk}\frac{dx^j}{dp}\frac{dx^k}{dp} = 0 \qquad \ldots(0.4)$$

of geodesics in terms of Π^l_{jk} uses the parameter p, and thus gives rise to projective invariant solutions.

Very recently, Kobayashi used the classical concept of projective invariance to show that every Riemannian manifold with Projective structure[11,16] carries an intrinsically defined pseudo-distance and Caratheodory-type pseudo-distance which turns out to be a distance if Ricci-tensor is bounded. He obtains these results by using the techniques of complex-manifold theory more specifically by using projective maps (see section 1) in place of holomorphic maps.

Since the notion of pseudo-distance (i. e. $d(x, y) \geqslant 0\ x \neq y,\ x, y \in M$) is relatively closer to space-time rather than to Riemannian space M the projective mapping should turn out to be equally (if not more) important in the case of space-time.

Moreover, an affine map is projective but not the other way round; thus projective maps are far more abundant than affine ones and, naturally, therefore, a study with them should yield a better pay off.

Now, a proper projective map ϕ: $(M, g) \to (M', g')$ which induces a nonvanishing 1-form ω on M, leads to the relation (see section 6):

$$\phi^* \operatorname{Ric}_{M'} = \operatorname{Ric}_M + d\omega - (n-1)\,\Box\omega.$$

*The L.H.S. of (0.2) denoted by $\{p,s\}$ is the Schwartzian derivative", in section 6 we use the symbol S_p for it.

Since Ricci-tensor is related to the presence of matter through the Einstein equation:

$$\text{Ric} - \tfrac{1}{2}Rg = 8\pi T$$

these maps can be associated with the creation or annihilation of matter[17].

In space-time manifolds where concept of completeness and singularity is much more complex and where singularity signifies (among other things such as unbounded curvature) the absence of complete (to be more precise b-complete) causal curves, it turns out that for Ricci-parallel manifolds projective parameter can by used to alter the incompleteness. It is for these reasons that the author pursued this study in spite of a vast amount of literature on complexities of Lorentzian manifolds, from modern and classical point of view[7,15,19,22,23].

Section 1 is devoted to the preliminaries. In section 2, we use projective maps to define the concept of projective equivalence in space-time (M, g) and show that the nature of causality is preserved under projective maps. In section 3 using projective parameter, we define a 5-dimensional manifold which imbeds (M, g) and show that it is an analogue of Kaluza-Klein manifold[9]. In section 4, we prove a singularity theorem with the help of our results, after obtaining a criterion for time completeness.

In section 5 we write the Jacobi equation in terms of projective parameter for hyperbolic manifolds and using the invariance of conjugate points under projective maps show that causal futures of two projectively related sets are isomorphic.

In section 6 we establish a Schwartzian lemma type result between two projectively related manifolds. Section 7 is devoted to examples.

1. Preliminaries

We choose a time-oriented space-time Lorentz manifold $(M, g) \equiv M$ i. e., a manifold whose metric g is locally reducible to $-1, +1, +1 +1$ and whose time-like vectors can be labelled as future-pointing or past-pointing.

The notations used are those of Hawking and Ellis[6].

A curve $x(t) = (x^i(t))$ (assumed to be regular) is time-like, null, space-like, or causal if $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} < 0$, $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$, $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} > 0$, or $g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \leqslant 0$, where indices i, j take the values 0, 1, 2, 3 and the coordinate x^0 corresponds to the time coordinate. The causal curves are also referred to as non-spacelike curves. A 3-dimensional submanifold imbedded in (M, g) is space-like if it admits a global time-like normal vector field[6].

Throughout this paper, we shall denote by S an arbitrary set of M and by $\mathscr{U}$ a convex normal neighborhood of some point q of M (see section 2 for definition).

The causal future of S relative to $\mathscr{U}$ denoted $J^+(S, \mathscr{U})$ is the union of $S \cap \mathscr{U}$ with the set of all points in $\mathscr{U}$ which can be reached from S by future directed non-space like curves in $\mathscr{U}$ (ref. 6).

When $\mathscr{U}$ is whole of M it is denoted $J^+(S)$. For a point $p \in M$ and $\mathscr{U} \neq M$ it is $J^+(p, \mathscr{U})$ and $\mathscr{U} = M$ it is $J^+(p)$. Similary the causal past of $p \in M$ is denoted by $J^-(p, \mathscr{U})$ or $J^-(p)$ according as $\mathscr{U} \neq M$ or $\mathscr{U} = M$.

A set $\mathscr{U}$ is causally convex if no non-space-like curve intersects $\mathscr{U}$ in a disconnected set. A point $p \in M$ is called strongly causal if p has arbitrarily small causally convex neighborhoods, and M is strongly causal if it is strongly causal at every point[1].

M is called globally hyperbolic if for every pair of points p, q in M the set $J^+(p) \cap J^-(q)$ is compact[1]. We say that causality condition holds on M if there are no closed non-space-like curves on M.

A vector field X along a geodesic $\gamma = (x(t))$ on M is a Jacobi field if it satisfies the Jacobi equation[12].

$$\nabla^2 X + R(X, \dot{x})\, \dot{x} = 0 \qquad \ldots(1.1)$$

where ∇ denotes the covariant derivation, the $R(y, z) = [\nabla_y, \nabla_z] - \nabla_{[y,z]}$ stands for the curvature operator and $\dot{x} = \dfrac{dx}{dt}$ for the tangent vector to γ.

Let Δ be the open interval $\{u: -1 < u < 1\}$ with Poincaré metric :

$$ds^2 = \frac{4du^2}{(1 - u^2)^2}.$$

Definition 1.1 – The mapping $f: \Delta \to M$ is called a projective mapping (by Kobayashi[10]) if f describes a geodesic in M and if the natural parameter U for Δ is a projective parameter for this geodesic.

We treat the above definition as the starting point for this study and obtain from it the classical equation (0.2) for projective parameter in section 3.

2. Projective Invariance of Causality

Let Aut (Δ) denote the group of automorphisms of the interval $\Delta = (-1 < u < 1)$. Elements of Aut (Δ)* denoted α are linear fractional transformations of Δ, thus for

*In section 6 they are denoted differently.

$U \in \Delta$, $\alpha(u) = \dfrac{au+b}{cu+d} \in \Delta$ where a, b, c, d are real numbers subject to $ad - bc \neq 0$, $\left|\dfrac{d}{c}\right| > 1$.

For the sake of distinction, we denote Δ and $\alpha(\Delta)$ by I and I' and their natural parameters by p and p'. We use Definition 1.1 to define :

Definition 2.1—Let S and S' be two diffeomorphic subsets of space-time (M, g) and let $f : I \to S$ and $f' : I' \to S'$ define projective maps into S and S'. The sets S and S' are projectively related by ϕ if the following diagram (Fig. 1)

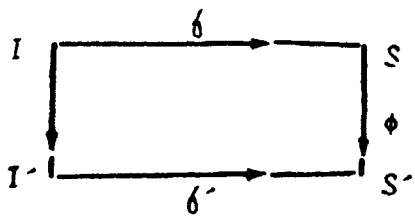

FIG. 1.

satisfies the equality $\phi \circ f = f' \circ \alpha$.

The above relation means that S and S' are in geodesic corrrespondence. We call the geodesic correspondence "proper", if the mapping α is proper.

Definition 2.2—The mapping $\alpha : I \to I'$ defined as $\alpha(p) = p' = \dfrac{ap+b}{cp+d}$ $ad - bc \neq 0$, $(a, b, c, d \in R)$ is called proper if:

(a) it is orientation preserving,

(b) $\dfrac{d\alpha}{dp} = \dfrac{dp'}{dp} \neq 0$, and

(c) $\dfrac{dp'}{dp}$ as well as $\dfrac{dp}{dp'}$ do not approach ∞ for any choice of $(a, b; c, d)$.

The condition (c) implies that $p \neq -\dfrac{d}{c}$ and $p' \neq \dfrac{a}{c}$.

The "proper" geodesic correspondence between S and S' can be locally described as follows: Given a coordinate system $x = (x^i)$ on S, the corresponding coordinate

system $x' = (x')$ on S is induced diffeomorphism $\phi : S \to S'$. A geodesic of S will be the solution of

$$\frac{d^2x^i}{dp^2} + \Gamma^i_{jk} \frac{dx^j}{dp} \frac{dx^k}{dp} = 0^*. \qquad \ldots(2.1)$$

From Definitions 2.1 and 2.2 a geodesic of S' will be obtained from (2.1) when p is replaced by $p' = \frac{ap + b}{cp + d}$ and x by x'. Thus using x' in place of x and observing that

$$\frac{dx^{i'}}{dp} = \frac{dx^{i'}}{dp'} \cdot \frac{dp'}{dp}$$

and

$$\frac{d^2x^{i'}}{dp^2} = \frac{d^2x^{i'}}{dp'^2} \left(\frac{dp'}{dp} \right)^2 + \frac{dx^{i'}}{dp'} \frac{d^2p'}{dp^2}$$

we have from (2.1) :

$$\left(\frac{d^2x^{i'}}{dp'^2} + \Gamma^{i'}_{j'k'} \frac{dx^{j'}}{dp'} \frac{dx^{k'}}{dp'} \right) \left(\frac{dp'}{dp} \right)^2 = - \frac{dx^{i'}}{dp'} \frac{d^2p'}{dp^2} .$$

A simple computation yields that this equation can be rewritten as

$$\frac{d^2x^{i'}}{dp'^2} + \Gamma^{i'}_{j'k'} \frac{dx^{j'}}{dp'} \frac{dx^{k'}}{dp'} = A(p') \frac{dx^{i'}}{dp'} \qquad \ldots(2.2)$$

where $A(p') = \frac{-2c}{(a - cp')}$ is a smooth function on I' in view of condition (c). Equation (2.2) is the differential equation of a geodesic in S'.

The following result shows that a 'proper' projective map preserves the nature of a geodesic.

Proposition 2.1—A mapping ϕ which defines proper geodesic correspondence between two subsets S and S' of (M, g) preserves the nature of geodesics, i.e. space-like and non-space-like geodesics are mapped into space-like and non-space-like geodesics respectively.

PROOF : Let $\mathscr{C} : x(p) = ((x^i(p))$ be a geodesic in S and $\phi(\mathscr{C}) = \mathscr{C}' : x'(p') = (x^{i'}(p'))$ in S'. The relation $x^{i'} = \phi^{i'}(x^i)$ where $i, i' = 0, 1, 2, 3$ is the coordinate expression for ϕ.

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*Γ^i_{jk} in eqn. (2.1) and (2.2) are classical π^i_{jk} (i.e. those given by 0.3). This notation is justifiable in view of the result established in section 3 (3.17, 3.18).

Denote the fundamental tensor in x' and x-system by $g_{i'j'}$ and g_{ij} respectively and recall that

$$\text{i)} \quad g_{i'j'} = g_{kl} \frac{\partial x^k}{\partial x^{i'}} \frac{\partial x^l}{\partial x^{j'}}$$

$$\text{ii)} \quad dx^{i'} = \frac{\partial x^{i'}}{\partial x^m} dx^m \text{ and}$$

$$\text{iii)} \quad dx^{j'} = \frac{\partial x^j}{\partial x^n} dx^n.$$

Consider the expression $g_{i'j'} \frac{dx^{i'}}{dp'} \frac{dx^{j'}}{dp'}$ and the successive transformations of if by using (i), (ii), and (iii). It is easy to see that :

$$g_{i'j'} \frac{dx^{i'}}{dp'} \frac{dx^{j'}}{dp'} = \left(g_{kl} \delta^k_m \cdot \delta^l_n \right) \frac{dx^m}{dp} \frac{dx^n}{dp} \left(\frac{dp}{dp'} \right)^2$$

$$= g_{mn} \frac{dx^m}{dp} \frac{dx^n}{dp} \left(\frac{dp}{dp'} \right)^2.$$

Since $\left(\frac{dp}{dp'} \right)^2 = \frac{(ad - bc)^2}{(cp' - a)^4} > 0$ it follows that if $g_{mn} \frac{dx^m}{dp} \frac{dx^n}{dp} > 0$ then same is true of $g_{i'j'} \frac{dx^{i'}}{dp'} \frac{dx^{j'}}{dp'}$; conversely if $g_{i'j'} \frac{dx^{i'}}{dp'} \frac{dx^{j'}}{dp'} > 0$ then $g_{mn} \frac{dx^m}{dp} \frac{dx^n}{dp}$ is also > 0. Similar argument will work when $g_{mn} \frac{dx^m}{dp} \frac{dx^n}{dp}$ is < 0 or equals 0. Hence $\mathscr{C}'$ is non-space-like (space-like) geodesic if and only if $\mathscr{C}$ is non-space-like (space-like).

An immediate consequence of above proposition is the following result.

Proposition 2.2—Let S and S' be 4-dimensional subsets of (M, g) which satisfy the hypothesis of Proposition 2.1. Corresponding to every space-like 3-surface $\mathscr{H}$ imbedded in S there exists a unique space-like 3-surface $\mathscr{H}'$ imbedded in S'.

The surface $\mathscr{H}'$ is the projective image of $\mathscr{H}$.

PROOF : Let the space-like 3-surface $\mathscr{H}$ be defined by the imbedding $\psi : \mathscr{H} \to S$. By definition the set $(\psi(\mathscr{H}))^\perp$ is the one-dimensional space spanned by the congruence of time-like geodesics. Thus for a point $p \in \mathscr{H}$ there exists a non-zero vector $\mathbf{n} \in T^*_{\psi(p)}$ such that for every vector X in the tangent space T_p, one has $< \mathbf{n}, \psi_* (x) > = 0$ where $<, >$ is the inner product in S.

The vector $\mathbf{n}$ is unique up to a sign and a normalizing factor. If the 3-surface $\mathscr{H}$ is locally given by $F = 0$, then $\mathbf{n}$ may be taken as $dF \neq 0$.

It is known that components $\frac{\partial F}{\partial x^i} = F_{,i}$ satisfy $g^{ij} F_{,i} F_{,j} < 0$ and, hence, define a time-like geodesic orthogonal to $\mathscr{H}$.

In view of Proposition 2.1, a time-like geodesic of S is mapped to a time-like geodesic in S'. In particular, components $F_{,i}$ are mapped to $F'_{,i'}$ where

$$F'_{,i'} = F_{,i} \frac{\partial x^i}{\partial x^{i'}}. \qquad \ldots(2.1)$$

We assume further that the time-like geodesic-congruence $(\psi(H))^{\perp}$ is free of conjugate points, whence the congruence $\phi((\psi(\mathscr{H}))^{\perp})$ is free of conjugate points as well (a fact to be proved later in Proposition 5.1).

This observation along with the Proposition 2.1 implies that the set: $\{\phi(\psi(p)) \mid p \in \mathscr{H}\} \subset \mathscr{S}'$ must be a space-like 3-surface whose local expression from equality (2.1) is $F' = \psi \circ F = 0$. The imbedding $\psi' : \mathscr{H}' \to S'$ is uniquely determined by the composite mapping $\phi \circ \psi$.

It should be noted that in tho case of the above propositions we have considered S and S' to be subsets of the same space-time (M, g). The results would, however, hold even if they were subsets of two distinct space-times (M,g), (M', g') as long $\phi : \mathscr{S} \to S'$ satisfied the requisite conditions.

Our next result is valid only when S and S' are subsets of distinct space-times.

Let (M', g) and (M', g') be two space-times which contain projectively-related diffeomorphic subsets S and S'; i. e., there exists a map $\phi : S \to S'$ which satisfies the equality $\phi \circ f = f' \circ \alpha$ with reference to Fig. 2.1. Then we have·

Proposition 2.3—If $S \subset M$ satisfies the causality condition and ϕ is proper, then $S' \subset M'$ also satisfies the causality condition.

Corollary 2.4—Let ϕ define a diffeomorphism between M and M' and let S be the chronological future $I^+(x)$ or the causal future $J^+(x)$ of a point x of M, such that $\phi|_S$ satisfies the conditions of Fig. 2.1 in relation to its image $\phi(S) = S'$, then S' is the chronological future or the causal future (respectively) of the point $x' = \phi(x) \in M'$. Proof of Proposition 2.3 and Corollary 2.4 is a consequence of following simple lemmas which characterise a proper projective map ϕ defined on space-time (M, g) or on a subset S of it.

Lemma 2.5—The map ϕ carries simply convex subsets to simply convex subsets.

Lemma 2.6—The map ϕ carries simple regions to simple regions.

Lemma 2.7—The map ϕ preserves the chronology.

To begin with, we elaborate on the concept of causality condition of M and give some definitions. Recall that a causal trip from x to y denoted $x \prec y$ $(x, y \in M)$ is a piecewise future-oriented causal geodesic which may possibly be degenerate. This means that x and y are joined by a sequence of time-like or null geodesic arcs each of which has an end point in future direction[14]. A curve $\mathscr{C}$ is a causal curve if and only if for all $a, b \in \mathscr{C}$ and for every open set Q containing the portion of $\mathscr{C}$ which joins a to b there is a causal trip from a to b which lies entirely in Q.

M (or a subset S of M) is said to satisfy the causality condition if it can be covered by a net of causal curves none of which is closed, i. e. there do not exist two distinct points x and y in M (or in S) for which $x < y$. and $y < x$ is simultaneously true.

Definition 2.3—Let Q be a star-shaped neighborhood of the origin in $T_q(M)$ for $q \in M$, the set $\exp_q(Q)$ is called a convex normal neighborhood of q (ref. 14).

Definition 2.4—Let U denote that convex normal neighborhood $\exp_q(Q)$ of q which is also a normal neighborhood of any other point $p \in U$, then it is called a simply convex neighborhood (subset) of M (ref. 14).

Proof of Lemma 2.5 : We recall that any two points p, q of a simply convex subset U of M can be joined by a unique geodesic lying within it[14]. Given $U = \exp_q(Q)$ and the mapping ϕ the following diagram (Fig. 2) is well defined :

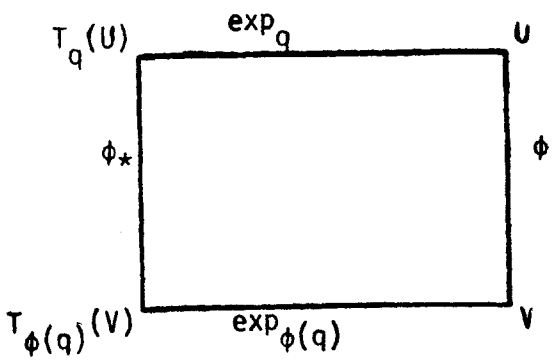

FIG. 2.

which determines the simply convex subset V. Since ϕ carries geodesics to geodesics, the unique geodesic joining p, q in U will be mapped to the geodesic joining $\phi(p)$, $\phi(q)$ in V.

Proof of Lemma 2.6 : By definition, a simple region N of M is a simply convex open subset of M whose closure $\bar{N}$ is compact and is contained in a simply convex open set. Also any two points $p, q \in \bar{N}$ can be joined by a unique geodesic in N.

From the above lemma it is evident the $\phi(N)$ must be a simple region.

Proof of Lemma 2.7 : A point $x \in S \subset M$ is said to chronologically precede another point y of S, if it can be joined to x by picewise future-oriented time-like geodesics. Symbolically we write it as $x \ll y$.

The set $I^+(x) \mid S = \{y \in \mathscr{S} \mid x \ll y\}$ is called the chronological future of x relative to $\mathscr{S}$. The set $I^+(x)$ is known to be an open set. Since ϕ carries time-like geodesics to time-like geodesics and preserves the orientation, the chronology in the set $\phi(S)$ is preserved under ϕ. More explicitly $x \ll y$ implies $\phi(x) \ll \phi(y)$, and $\phi(I^+(x)) = I^+(\phi(x))$.

Remark : It is known[14] that $x \ll y$ implies $x < y$ hence it follows that $\phi(x) \ll \phi(y)$ will also imply $\phi(x) < \phi(y)$.

Proof of Proposition 2.3 : We are given that $S \to S'$ is a proper projective mapping where S satisfies the causality condition.

Let x and y be two points S such that $x < y$. By the definition of causal trip we have a sequence of points $\{x_i\}$ $i = 1, 2, \ldots n$ with $x = x_0$ and $y = x_n$ such that each pair x_{i-1}, x_i can be joined by a future-oriented causal geodesic. Suppose that this causal geodesic segment is given by $f : \mathscr{T}_i \to S$ where $\mathscr{T}_i$ is the unit interval with p_i as projective parameter and $f(0) = x_{i-1} < x_i = f(1 - \epsilon)$. By hypothesis this geodesic segment is not closed or degenerate. Let the map under ϕ of this geodesic segment be given by $f' : \mathscr{T}_i' \to S'$. It is easy to check that it is not closed. Suppose that this is not the case, then $f'(0)$ must be equal to $f'(1 - \epsilon)$. But 0 and $1 - \epsilon$ belonging to $\mathscr{T}_i'$ are the automorphic images $\alpha(p_{i_1})$ and $\alpha(p_{i_2})$ for two distinct elements p_{i_1} and p_{i_2} belonging to $\mathscr{T}_i$.

$$f(p_{i_1}) = \phi^{-1}(f'(\alpha(p_{i_1}))) = \phi^{-1}(f'(0)) = \phi^{-1}(f'(1 - \epsilon))$$

$$= \phi^{-1}(f'(\alpha(p_{i_2}))) = f(p_{i_2})$$

since ϕ is a diffeomorphism.

But $(f(p_{i_1}) \neq f(p_{i_2})$ because of our assumption on the geodesic segment $[x_{i-1}, x_i]$. Moreover ϕ preserves the orientation, therefore, we have $\phi(x_{i-1}) < \phi(x_i)$. This is true for all i, hence using the result that $x_1 < x_2$, $x_2 < x_3$ implies $x_1 < x_3$, we obtain that $x' < y'$, or that x', y' are joined by future oriented causal curve. In

order to show that causality is not violated on S' we must ensure that the causal curve can not be extended to a closed time-like curve in S' beyond the simple region containing the points x', y'. Suppose that it could be extended into a closed time-like curve in S', since ϕ is a diffeomorphism this would mean in turn that there exists a closed time-like curve in S.

Proof of corollary 2.4 is immediate from the above proposition and from Lemma 2.7.

In view of the above results we have another simple but important result. As the proof is rather routine, we state it without proof.

Proposition 2.8—Let (M,g) be a globally hyperbolic space-time and let $\phi : (M, g) \to (M', g')$ be a proper projective mapping then (M', g') is globally hyperbolic.

3. Gauge Variable and the Ambient Manifold for Space-Time

The study in this section is motivated by the Kaluza-Klein[9] theory[9] of 5-dimensional spaces. We use the gauge variable induced by the projective parameter p to define a 5-dimensional manifold M^* which imbeds the space-time (M, g) and show the similarity between M^* and Kaluza-Klein manifold.

We introduce the gauge-variable :

$$x^4 = -\frac{1}{2\lambda} \log \frac{ds}{dp} \quad * \qquad \ldots(3.1)$$

on every geodesic

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad \ldots(3.2)$$

determined by Γ (ref. 2).

To begin with, we obtain the transformation rule for the gauge variable in the event of change from one coordinate system to another.

Let $\bar{s}$ denote the affine parameter of the geodesic for projectively related connection $\bar{\Gamma}$ [see (0.1)] Equation (3.2) gives:

$$\frac{d^2x^i}{d\bar{s}^2} + \bar{\Gamma}^i_{jk} \frac{dx^j}{d\bar{s}} \frac{dx^k}{d\bar{s}} = 0. \qquad \ldots(3.3)$$

But expressing

*The gauge variable was introduced by Veblen in 1922. Hlavaty's[8] gauge variable differs in scale from the one used by Veblen[24] and Berwald[3] and in this paper.

$\frac{dx^i}{d\bar{s}}$, $\frac{d^2x^i}{d\bar{s}^2}$ in terms of $\bar{s}$, and replacing Γ by $\bar{\Gamma}$ in equation (3.2) it becomes:

$$\frac{d^2x^i}{d\bar{s}^2} \cdot \left(\frac{d\bar{s}}{ds}\right)^2 + \frac{dx^i}{d\bar{s}} \frac{d^2\bar{s}}{ds^2} + \left(\bar{\Gamma}^i_{jk} - \delta^i_j p_k - \delta^i_k p_j\right) \times \frac{dx^j}{d\bar{s}} \frac{dx^k}{d\bar{s}} \left(\frac{d\bar{s}}{ds}\right)^2 = 0. \quad \ldots(3.4)$$

Since (3.3) and (3.4) are same, we have :

$$\left(\left(\frac{d^2\bar{s}}{ds^2}\right) \Big/ \left(\frac{d\bar{s}}{ds}\right)^2\right) \frac{dx^i}{d\bar{s}} - 2p_j \frac{dx^i}{d\bar{s}} \frac{dx^j}{d\bar{s}} = 0$$

or

$$\frac{d^2\bar{s}}{ds^2} - 2p_j \frac{dx^j}{ds} \frac{d\bar{s}}{ds} = 0.$$

This lead to

$$\frac{d\bar{s}}{ds} = e^{2\int p_j dx^j}. \quad \ldots(3.5)$$

Thus denoting by $\bar{x}^4$ the gauge variable w. r. to $\bar{s}$ we have :

$$\bar{x}^4 = x^4 - \frac{1}{\lambda} \int_a^x p_l \, dx^l + C. \quad \ldots(3.6)$$

As a consequence, a coordinate transformation

$$\bar{x}^i = \bar{x}^i(x^0, x^1, x^2, x^3), J \equiv \frac{\partial(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)}{\partial(x^0, x^1, x^2, x^3)} \neq 0 \quad \ldots(3.7)$$

on M can be extended to the transformation :

$$\bar{x}^4 = \bar{x}^4 + \Phi(x^0, x^1, x^2, x^3)$$

$$\bar{x}^i = \bar{x}^i(x^0 x^1, x^2, x^3), J \neq 0 \quad \ldots(3.8)$$

provided $p_l \, dx^l$ is an exact differential.

Define point functions

$$\Gamma^i_{jk} = \Gamma^i_{kj}$$

on geodesics of M, which satisfy the differential equation :

$$\{p, s\} = 2\lambda \ \Gamma^i_{jk} \ \frac{dx^j}{ds} \frac{dx^k}{ds} \qquad \ldots(3.9)$$

λ being an arbitrary constant.

If p and s are assumed to be invariant under coordinate transformations, then Γ^i_{jk} are seen to be components of a covariant tensor of order two*.

We note that in view of identity

$$\{p, s\} = -\{s, p\} \Big/ \left(\frac{ds}{dp} \right)^{2*} \qquad \ldots(3.10)$$

eqn. (3.9) can be written as :

$$\{s, p\} = 2\lambda \ \Gamma^i_{jk} \ \frac{dx^j}{dp} \frac{dx^k}{dp} . \qquad \ldots(3.11)$$

Also treating p as an independent parameter we write eqn. (3.2) as :

$$\left(\frac{d^2x^i}{dp^2} - \frac{dx^i}{dp} \frac{d^2s}{dp^2} \right) \Big/ \left(\frac{ds}{dp} \right)^2 + \left(\Gamma^i_{jk} \frac{dx^j}{dp} \frac{dx^k}{dp} \right) \Big/ \left(\frac{ds}{dp} \right)^2 = 0 \qquad \ldots(3.12)$$

after using

$$\frac{d}{ds}\left(\frac{dx^i}{ds} \right) = \frac{d}{dp}\left(\frac{dx^i}{dp} \Big/ \frac{ds}{dp} \right) \Big/ \left(\frac{ds}{dp} \right)$$

$$= \frac{d^2x^i}{dp^2} \Big/ \left(\frac{ds}{dp} \right)^2 - \frac{dx^i}{dp} \frac{d^2s}{dp^2} \Big/ \left(\frac{ds}{dp} \right)^3 .$$

From (3.1) we have on differentiation :

$$\frac{dx^i}{dp} = -(2\lambda)^{-1} \left(\frac{d^2s}{dp^2} \Big/ \left(\frac{ds}{dp} \right) \right). \qquad (3.13)$$

We use (3.13) to eliminate $\frac{d^2s}{dp^2}$ from (3.12), thus :

$$\frac{d^2x^i}{dp^2} + 2\lambda \ \frac{dx^i}{dp} \frac{dx^s}{dp} + \Gamma^i_{jk} \ \frac{dx^j}{dp} \frac{dx^k}{dp} = 0. \qquad \ldots(3.14)$$

*See Appendix 3.

*See Appendix 1.

Again since

$$\frac{d^2x^4}{dp^2} = -\frac{1}{2\lambda}\left(\frac{d^3s}{dp^3}\Big/\left(\frac{ds}{dp}\right) - \left(\frac{d^2s}{dp^2}\right)^2\Big/\left(\frac{ds}{dp}\right)\right) \qquad ...(3.15)$$

we write the Schwartzian derivative $\{s, p\}$ in terms of x^4 and p :

$$\{s, p\} = \left(\frac{d^3s}{dp^3}\right)\Big/\frac{ds}{dp} - \frac{3}{2}\left(\frac{d^2s}{dp^2}\right)\Big/\left(\frac{ds}{dp}\right)^2$$

$$= -2\lambda\frac{d^2x^4}{dp^2} + \left(\frac{d^2s}{dp^2}\right)^2\Big/\left(\frac{ds}{dp}\right)^2 - \frac{3}{2}\left(\frac{d^2s}{dp^2}\right)^2\Big/\left(\frac{ds}{dp}\right)^2$$

$$= -2\lambda\frac{d^2x^4}{dp^2} - \frac{1}{2}\left(-2\lambda\frac{dx^4}{dp}\right)^2$$

$$= -2\lambda\left[\frac{d^2x^4}{dp^2} + \lambda\left(\frac{dx^4}{dp}\right)^2\right].$$

From (3.11) we therefore have

$$\frac{d^2x^4}{dp^2} + \lambda\left(\frac{dx^4}{dp}\right)^2 + \Gamma^4_{jk}\frac{dx^j}{dp}\frac{dx^k}{dp} = 0. \qquad ...(3.16)$$

Let Greek and Latin indices stand for 0, 1, 2, 3, 4 and 0, 1, 2, 3, respectively, then using

$$\text{(a)}\quad \Pi^\alpha_{jk} = \Gamma^\alpha_{jk}$$

$$\text{(b)}\quad \Pi^\alpha_{\beta 4} = \Pi^\alpha_{4\beta} = \lambda\delta^\alpha_\beta \qquad ...(3.17)$$

we can write the eqns. (3.34) and (3.16) compactly as :

$$\frac{d^2x^\alpha}{dp^2} + \Pi^\alpha_{\beta\gamma}\frac{dx^\beta}{dp}\frac{dx^\gamma}{dp} = 0. \qquad ...(3.18)$$

Equation (3.11) defines the geodesics of 5-dimensional manifold M^* for which coordinate-transformations are given by (3.8). Hence we have shown :

Proposition 3.1—If space-time (M, g) admits a 1-form given locally $p_i\,dx^i$ say, then there exists a 5-dimensional manifold M unique up to projective transformations which embeds M in such a manner that the connection coefficients of M^* and M are related by (3.18). In addition to this, they satisfy $\frac{\partial\Pi^\alpha_{\beta\gamma}}{\partial x^4} = 0$.

We next show that in a certain sense M^* is a generalisation of Kaluza-Klein manifold[9].

We recall that a Kaluza-Klein manifold denoted M^5 can be thought of as a principal fiber bundle $P(M^4, SO(2))$ over a space-time manifold M^4 with structural (gauge) group $SO(2)$. M^5 is homeomorphic to topological product $M^4 \times R$ and is endowed with $SO(2)$ invariant metric, given locally as :

$$g = g_{\mu\nu}\, dx^\mu\, dx^\nu \pm (dx^4 + A_\mu\, dx^\mu)^2 \qquad (\mu, \nu = 0, 1, 2, 3) \quad \ldots(3.19)$$

$A_\mu = \dfrac{g_{\mu 4}}{g_{44}}$ being the gauge potential which satisfies

$$A'_{\mu'} = A_\mu - \frac{\partial \lambda}{\partial x^\mu} \qquad \ldots(3.20)$$

for an arbitrary scalar λ.

From (3.20) it follows that the field defined by

$$\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \equiv F_{\mu\nu} \qquad \ldots(3.20a)$$

is a skew-symmetric gauge-invariant tensor, that represents the electro-magnetic field in the unifying field theory of Kaluza-Klein.

If we choose the + sign and $g_{44} = 1$ in (3.19) it can be written as:

$$g = g_{\mu\nu}\, dx^\mu\, dx^\nu + (dx^4 + g_{\mu 4}\, dx^\mu)^2. \qquad \ldots(3.21)$$

It is known that coordinate transformation on M^5 which satisfy (3.19) and (3.20) are of the form

$$x^{4'} = x^4 + \phi(x^0, x^1, x^2, x^3) \qquad \ldots(3.22a)$$

$$x^{\mu'} = x^{\mu'}(x^0, x^1, x^2, x^3)$$

with non-vanishing Jacobian $J = \dfrac{\partial (x^{0'}, x^{1'}, x^{2'}, x^{3'})}{\partial (x^0, x^1, x^2, x^3)}$. $\qquad \ldots(3.22b)$

Treating (3.20) (3.21) and (3.22) as characteristic properties of Kaluza-Klein manifold and using i, j in place of μ, ν we can enunciate the following result :

Proposition 3.2—The 5-dimensional manifold M^* obtained by using the projective parameter p can be identified through affine-connection with Kaluza-Klein mani-

fold M^5 (save for one connection parameter Γ^4_{44}) provided the electromagnetic field F_{ij} of M^5 satisfies :

$$\sum_j G^{4j} F_{ij} = 0; \; \Sigma_k G^{kj} F_{ik} = 0; \; \Sigma G^{ij} F_{ij} = 8\lambda \qquad \ldots(3.23)$$

where (3.21) has been written as $ds^2 = G_{\alpha\beta}\, dx^\alpha\, dx^\beta$.

PROOF : In the metric $ds^2 = G_{\alpha\beta}\, dx^\alpha dx^\beta$, $G_{44} = 1$, $G_{ii} = \{(g_{i4})^2 + g_{ii}\}$ $(i = 0.\ 1,2,3)$ and $G_{i\alpha} = g_{i\alpha}$ $(i \neq \alpha)$, also since A_μ of (3.19) is independent of x^4, $\frac{\partial g_{\alpha\beta}}{\partial x^4} = 0$, and this implies :

$$\text{(i)} \quad \frac{\partial G_{\alpha\beta}}{\partial x^4} = 0$$

$$\text{(ii)} \quad \frac{\partial G^{\alpha\beta}}{\partial x^4} = 0 \qquad \ldots(3.24)$$

where

$$G^{\alpha\gamma}\, G_{\beta\gamma} = \delta^\alpha_\beta \; .$$

Since coordinate transformations (3.8) and (3.22) of M^* and M^5 are the same to prove our result we have to show that metricity condition $G_{\alpha\beta,\gamma} = 0$ is compatible with

$$\frac{\partial \Pi^\alpha_{\beta\gamma}}{\partial x^4} = 0 \qquad \ldots(3.25)$$

which is one of the defining equations of M^*, and that (3.20a) together with $G_{\alpha\beta,\gamma} = 0$, lead to (3.23).

To use $\Pi^\alpha_{\beta 4} = \lambda\delta^\alpha_\beta$ we write $\Pi^\alpha_{\beta 4} = \frac{1}{2}\, G^{\alpha\eta}\, \{\beta 4, n\}$

where $\{\alpha\beta,\gamma\}$ denotes the Christoffel connections of first kind with respect to G. Thus

$$\Pi^\alpha_{\beta 4} = \frac{1}{2}\, G^{\alpha\eta} \left(\frac{\partial G\beta_\eta}{\partial x^4} + \frac{\partial G_{4\eta}}{\partial x^\beta} - \frac{\partial G_{\beta 4}}{\partial x^\eta} \right).$$

We make the following choices for α and β for obvious reasons :

$$\Pi^4_{i4} = \frac{1}{2}\, G^{4n} \left(\frac{\partial G_{4\eta}}{\partial x^i} - \frac{\partial G_{i4}}{\partial x^\eta} \right)$$

(equation continued on p. 645)

$$= \frac{1}{2} G^{44} \left(\frac{\partial G_{44}}{\partial x^i} - \frac{\partial G_{i4}}{\partial x^4} \right) + \frac{1}{2} G^{4j} \left(\frac{G_{4j}}{\partial x^i} - \frac{G_{i4}}{\partial x^j} \right)$$

$$= \frac{1}{2} G^{4j} F_{ij}$$

$$= 0$$

$$\Pi^i_{i4} = \frac{1}{2} G^{i\eta} \left(\frac{\partial G_{4\eta}}{\partial x^i} - \frac{}{\partial x^\eta} \right)$$

$$= \frac{1}{2} G^{i4} \left(\frac{\partial G_{44}}{\partial x^i} - \frac{\partial G_{i4}}{\partial x^4} \right) + \frac{1}{2} \sum_{j \neq i} G^{ij} \left(\frac{\partial G_{4j}}{\partial x^i} - \frac{\partial G_{i4}}{\partial x^j} \right)$$

$$= \frac{1}{2} \sum_{j \neq i} G^{ij} F_{ij}$$

$$= 4\lambda$$

$$\Pi^j_{i4} = \frac{1}{2} G^{j\eta} \left(\frac{\partial G_{4\eta}}{\partial x^i} - \frac{\partial G_{i4}}{\partial x^\eta} \right) \quad i \neq j$$

$$= \frac{1}{2} G^{j4} \left(\frac{\partial G_{44}}{\partial x^i} - \frac{\partial G_{i4}}{\partial x^4} \right) + \frac{1}{2} G^{jk} \left(\frac{\partial G_{4k}}{\partial x^i} - \frac{\partial G_{i4}}{\partial x^k} \right)$$

$$= \frac{1}{2} \sum_k G^{jk} F_{ik}$$

$$= 0 \qquad \text{since } i \neq j.$$

Computation of Π^4_{44} shows that it is zero in terms of G_{ij}, whereas it is λ in the case of M^*.

In order to check (3.25) we have to show that $\frac{\partial}{\partial x^4} (G^{\alpha\eta} \{\beta\gamma, n\})$ is zero, i.e.

$$\frac{\partial}{\partial x^4} (G^{\alpha\eta}) \{\beta\gamma, n\} + G^{\alpha\eta} \frac{\partial}{\partial x^4} \left(\frac{\partial G_{\beta\eta}}{\partial x^\gamma} + \frac{\partial G_{\gamma\eta}}{\partial x^\beta} - \frac{\partial G_{\beta\gamma}}{\partial x^\eta} \right) = 0.$$

Now smoothness of $G_{\alpha\beta}$ gives the vanishing of the second term, while the first term, is zero from (3.24) (ii).

It may be recalled that projections of geodesics of Kaluza-Klein manifold M^5 on M are the world lines of charged particles given by the equations :

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = \frac{e}{m} F^i_j \frac{dx^j}{ds} \qquad \ldots(3.26)$$

where

$$F^i_j = G^{4k} F_{kj} \qquad \ldots(3.27)$$

and m, e are the mass and the charge of the particle.

Above discussions lead to two important remarks of physical interest:

Remark 3.3 : From (3.25) on M^* and (3.20a) and (3.24) on M^5 it follows that unlike Kaluza-Klein field a gauge invariant field on M^* can not be related to a metric, but like Yang-Mills field can be related to an affine connection.

In the broader sense, however, since Kaluza-Klein theory and Yang-Mills theory with same underlying structure $P(M^4, SO(2)$ may be viewed as similar through Ehresmann connection in $P(M_4, SO(2)$ we conclude that M^* and $M^5 \cong M^4 \times R$ are locally same inspite of above remark.

Remark 3.4 : For $x^4 =$ constant, equation (3.18) gives the geodesics in M which in view of (3.26) can be interpreted as representing the trajectories of particles with no electric charge, in particular the world line of photons.

We next consider the curvature tensors on M^* and M to obtain projectively invariant curvature tensor.

Let ${}^*R^{\alpha}_{\beta\gamma\delta}$ and R^l_{jkl} denote the components of curvature tensor in M^* and M. It can be verified that ${}^*R^{\alpha}_{\beta\gamma\delta} = 0$ except for ${}^*R^4_{ijk}$ and ${}^*R^l_{jkl}$. The latter satisfies

$${}^*R^l_{jkl} = R^l_{jkl} + \lambda \left(\Gamma^4_{ik} \delta^l_l - \Gamma^4_{jl} \delta^l_k \right). \qquad (3.28)$$

Using (0.1) we can express the curvature tensor $\bar{R}^l_{jkl}$ for $\bar{\Gamma}$ as :

$$\bar{R}^l_{jkl} = R^i_{jkl} - (p_{j,k} - p_j p_k)\, \delta^l_l + (p_{j,l} - p_j p_l)\, \delta^l_k + p_{kl}\, \delta^l_j \qquad \ldots(3.29)$$

where $p_{j,k}$ denotes the covariant differentiation with respect to Γ and p_{kl} stands for $p_{k,l} - p_{l,k} = \dfrac{\partial p_k}{\partial x_l} - \dfrac{\partial p_l}{\partial x_k}$.

From (3.29) and (4. App3) we observe that the curvature tensor ${}^*\bar{R}^l_{jkl}$ and ${}^*\bar{R}^l_{jkl} = {}^*\bar{R}_{jk}$ for projectively related connection Γ satisfy :

$$*\bar{R}^i_{jkl} = *R^i_{jkl} - \frac{1}{2} p_{jk}\,\delta^i_l + \frac{1}{2} p_{jl}\,\delta^i_k + p_{kl}\,\delta^i_j \quad \ldots(3.30)$$

and

$$*\bar{R}_{jk} = *R_{jk} - \frac{5}{2} p_{jk}. \quad \ldots(3.31)$$

Equations (3.30) and (3.31) lead to projectively invariant tensor on M^* :

$$*B^i_{jkl} = *R^i_{jkl} + \frac{1}{5}\left[- *R_{[jk]}\,\delta^i_l + *R^{[}_{jl]}\,\delta^i_k + 2*R_{[kl]}\,\delta^i_j \right]$$

where $*R_{[jk]}$ denotes the skew symmetric part of $*R_{jk}$.

If we use (3.28) and its contraction :

$$*R_{jk} = R_{jk} + 3\lambda\ \Gamma^4_{jk} \quad \ldots(3.32)$$

we can write (3.32) as :

$$*B^i_{jkl} = R^i_{jkl} - \left(\frac{1}{5} R_{[jk]} - \lambda\Gamma^4_{jk}\right)\delta^i_l + \left(\frac{1}{5} R^{[}_{jl]} - \lambda\,\Gamma^4_{jl}\right) \times \delta^i_k + \frac{2}{5} R_{[kl]}\delta^i_j. \quad \ldots(3.33)$$

The contraction

$$*B^i_{jkl} = *B_{jk} = \frac{1}{2}\,(*R_{lk} + *R_{kj})$$

$$= \frac{1}{2}(R_{jk} + R_{kj}) + 3\lambda\Gamma^4_{jk}$$

is symmetric and its vanishing gives

$$\Gamma^4_{jk} = -\frac{1}{3\lambda}\frac{(R_{jk} + R_{kj})}{2}. \quad \ldots(3.34)$$

We eliminate Γ^4_{jk} etc. from (3.33) and obtain the projectively invariant Weyl tensor on M :

$$W^i_{jkl} = R^i_{jkl} + \frac{1}{5}(R_{kl} - R_{lk})\,\delta^i_j + \frac{1}{15}\left[(4R_j + R_{lj})\,\delta^i_k\right.$$

(equation continued on p. 648)

$$- (4R_{jk} + R_{kj}) \delta^i_i \Big]. \qquad \ldots(3.35)$$

Finally, from (3.9) and (3.34), we obtain

$$\{p, s\} = \frac{2}{3} R_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds}. \qquad \ldots(3.36)$$

This shows that if p is a projective parameter for a geodesic, then Schwartzian derivative can be expressed in terms of Ricci curvature of M.

Equation (3.36) in terms of arbitrary parameter t reads :

$$\{p, t\} = \frac{2}{3} R_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} \qquad \ldots(3.37)$$

which is a simple consequence of (3.App.3).

Thus starting with our definition of projective map we have come full circle. since eqn. (3.36) in classical literature is taken as the definition of projective parameter.

4. Projective Parameter and Singularity

Recall that a space-time (M, g) is singularity-free if it is b-complete (bundle-complete)[6]. The b-completeness implies g-completeness (geodesic-completeness). More explicitly, this says that there is an end point for each and every non-space-like geodesic as measured by an affine parameter.

Since affine and projective parameters s and p are related by the equation

$$\{s, p\} = - \left(\frac{ds}{dp} \right)^2 \{p, s\} \qquad \ldots(4.1)$$

it follows that (M, g) will be singularity-free if each and every non-space-like geodesic has an end point measured by a projective parameter. Another way of saying it, would be that every non-space-like geodesic can be extended to an arbitrary value of the parameter.

We mentioned in the introduction that projective maps can be used to alter the incompleteness into completeness. We elaborate now on this point.

Note that the projective map $f : (-1 < u < l) \to (M,g)$ introduces the parameter p on the geodesic which satisfies the equation

$$\{p, s\} = \frac{2}{3} R_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds}$$

in relation to the symmetric part R_{ij} of Ricci curvature on (M, g).

Suppose that space-time (M, g) carries a connection, such that R_{ij} is parallel; then obviously three cases can arise :

(i) $\{p, s\} = \eta^2$

(ii) $\{p, s\} = -\eta^2$

(iii) $\{p, s\} = 0$

where η is a constant greater than zero. Solutions to (i), (ii) and (iii) are (see Appendix 2)

$$\text{(i')} \quad p = \beta \tan(\sqrt{2}\,\eta s + \alpha)/2 + \gamma, \ \beta \neq 0$$

$$\text{(ii')} \quad p = \beta\,(1 - \alpha \exp(\sqrt{2}\eta s))^{-1} + \gamma, \qquad \alpha, \beta \neq 0$$

$$\text{(iii')} \quad p = \beta\,(s + \alpha)^{-1} + \gamma. \qquad \ldots(4.2)$$

From both (i') and (iii') it follows that as s moves in a certain interval, p moves from $-\infty$ to ∞.

We illustrate our point by using the following example of Geroch[5] in 2 dimensional Lorentzian manifold.

Example 4.1 (ref. 1)—Given Minkowski 2-space $(\mathbf{R}^2, g_0)$ with global coordinates (x, t) and metric $g_0 = ds^2 = dx^2 - dt^2$, use the conformal metric $g = \phi g_0$, where $\phi : \mathbf{R}^2 \to (0, \infty)$ is a smooth function with the following properties :

(1) $\phi(x, t) = 1$ for $x \leqslant -1$ and $x \geqslant 1$

(2) $\phi(x, t) = \phi(-x, t)$

(3) $\phi(0, t) \sim \frac{1}{t^4}$ as $t \to \infty$.

By property (2) the reflection : $(x, t) \to (-x, t)$ is an isometry of $(\mathbf{R}^2, g)$, since the fixed point set of an isometry is totally geodesic the t-axis can be parametrized as a time-like geodesic. By property (3) of ϕ, this geodesic is incomplete as $t \to \infty$.

Since Ricci-curvature is zero for $(\mathbf{R}^2, g)$, we can express the affine parameter (which is t) in terms of p by using (iii), and ϕ is seen to $\sim p^4$ as $t \to \infty$. Thus we note that as t (representing the time-axis) $\to \infty$, ϕ does not tend to zero, it tends to ∞ instead.

The above example implies :

Remark 4.2: If space-time (M, g) carries a parallel Ricci-tensor which is either zero or defines a positive definite form, then p maps the whole line R on time-like geodesics of (M, g).

Although the example that we choose pertains to only time-like curves, we observe that conclusions based on (i') and (iii') are valid for any type of geodesic.

From both (i') and (iii') it follows that as s moves in a certain interval, p moves from $-\infty$ to ∞. This means that there is a projective map from the entire line into the geodesic $x(t)$; this can happen only when M is geodesically complete. Hence we have shown :

Proposition 4.3—If geodesics of space-time M can be determined by using the projective parameter given by (i') or (iii') then M must be geodesically complete.

As a consequence of this result, we also have :

Corollary 4.4— (M, g) is singularity free if the projective parameter defining its geodesics can be expressed as a function of t given in (i') and (iii').

5. Projective Maps and Conjugate Points

Since a time-like geodesic is not maximal beyond the first conjugate point, we examine the projective invariance of a conjugate point in this section.

Now projective relatedness locally implies the existence of a projective connection [eqn. (0.3)] and it is known that among all affine connections given by (0.1) with the same projective connection there is a unique connection[14,20,21] (called normal affine connection) for which Π^i_{jk} coincides with affine connection Γ^i_{jk} .

As a consequence we observe that for any given 1-parameter family of geodesics parametrised by affine parameter s the Jacobi equation (1.1) can be writen in terms of p, i.e. as

$$\nabla_{\frac{d\gamma}{dP}} \nabla_{\frac{d\gamma}{dP}} J(p) + R\left(J(p), \frac{d\gamma}{dp}\right)\frac{d\gamma}{dp} = 0. \qquad \ldots(5.1)$$

We consider a one-parameter family of geodesics given by $\psi : (-\epsilon, \epsilon) \times [0, 1) \to (M, g)$ such that for fixed $\tilde{p}$ in $(-\frac{1}{2}\epsilon, \epsilon)$, $\psi(\tilde{p}, p)$ denoted by $\gamma^{\tilde{p}}(p)$ is a geodesic in (M, g) and for fixed p the curve $\gamma_p(\tilde{p})$ defines the separation curve for neighboring geodesics. The tangent vector $\frac{d}{d\tilde{p}}\gamma_p(\tilde{p})$ is the vector J in (5.1); for convenience of notations we shall use $\tilde{z}$ instead.

Using an orthonormal basis and a time-like geodesic family, eqn. (5.1) can locally be expressed as[6].

$$\frac{d^2\tilde{z}^i}{dp^2} = - R^i_{0j0}\,\tilde{z}^j \quad (i, j = 1, 2, 3). \qquad \ldots(5.2)$$

Two points P and Q on the geodesic $\gamma^0(p)$ are called conjugate if the vector $\tilde{z}$ which is not identically zero vanishes at these points.

Proposition 5.1 — Conjugate points of time-like geodesics are invariant under projective maps.

PROOF : Let S and S' be two subsets of (M', g) and (M', g') which are projectively related in the sense of §2. Let the one parameter-family of time-like geodesics in S given by $\psi : (-\epsilon, \epsilon) \times [0, 1) \to S$ be expressed by the triad $(\tilde{\gamma}_p(p), \tilde{\gamma}_p(p), \tilde{z})$, then, in view of Proposition 2.1, the corresponding triad for the one-parameter family of time-like geodesics in S' is given by $(\tilde{\gamma}_{p'}(p'), \tilde{\gamma}_{p'}(p'), \tilde{z}')$. The separation vector $\tilde{z}'$ of this time-like geodesic conguence equals $\frac{d}{d\tilde{p}'}\tilde{\gamma}_{p'}(p')$ and satisfies the Jacobi-equation:

$$\frac{d^2\tilde{z}'^i}{dp^2} = - R'^i_{0j'0}\,\tilde{z}^j \qquad \ldots(5.3)$$

where $R'^i_{0j'0}$ is the curvature tensor in terms of local coordinates x'^i of S'. Since S and S' are diffeomorphic, a choice of orthonormal frames which gives (5.3) is possible. From

$$\tilde{z} = \frac{d}{d\tilde{p}}(\tilde{p}_p(\tilde{\gamma})) = \frac{d}{d\tilde{p}}(\tilde{\gamma}_p(\tilde{p}))\frac{d\tilde{p}'}{dp'} = \frac{d\tilde{p}'}{d\tilde{p}}\left(\frac{d}{dp'}(\tilde{\gamma}_{p'}(\tilde{p}'))\right)$$

$$= \frac{ad - bc}{(c\tilde{p} + d)^2}\tilde{z}'$$

$$\tilde{z} = 0 \Leftrightarrow \tilde{z}' = 0.$$

Next, we consider the causal future $\mathcal{J}^+(S, \mathcal{U})$ of subset S of (M, g) relative to another set $\mathcal{U}$ of (M, g). Suppose S' is the subset obtain by using the infinitesimal point transformation:

$$'x^h = x^h + v^h\,\delta p$$

to carry the points of S.

If v^h are the components of a tangent vector to time-like geodesics and δp the projective distance along these curves is so small that there are no conjugate points in between, then we have the following :

Proposition 5.2—The causal future $\mathcal{J}^+(S, \mathcal{U})$ of S' relative to $\mathcal{U}$ is isomorphic to $\mathcal{J}^+(S, \mathcal{U})$, provided the set of those points of $\mathcal{J}^+(S', \mathcal{U})$ which can be reached from S by null geodesics is of measure zero.

PROOF : Let $\mathcal{P}'$ be an arbitrary point of S' which was point $\mathcal{P}$ in S, and let $\mathcal{J}^+(\mathcal{P}', \mathcal{U})$ be its causal future, i. e. the set of all points in $\mathcal{U}$ which can be reached from $\mathcal{P}'$ by future directed non-space-like curves in $\mathcal{U}$. If each of these curves is a time-like geodesic, then as $\mathcal{P}$ and $\mathcal{P}'$ lie along a time-like geodesic at infinitesimal distance $\mathcal{J}^+(\mathcal{P}, \mathcal{U})$ is isomorphic to $\mathcal{J}^+(\mathcal{P}', \mathcal{U})$.

In case the points of $\mathcal{J}^+(\mathcal{P}, \mathcal{U})$ are joined to $\mathcal{P}'$ by future directed non-space like curves that are not null geodesics, then it is known that these curves can be deformed into time-like curves, accordingly we have the same conclusion as above. Moreover, the set of null geodesics joining $\mathcal{P}'$ to $\mathcal{P}$ is of measure zero, and as such, the set of points $\mathcal{J}^+(\mathcal{P}', \mathcal{U})$ and $\mathcal{J}^+(\mathcal{P}, \mathcal{U})$ are same. Since $\mathcal{P}'$ was any point of S' we have proved the results.

6. PROJECTIVE MAPS AS DISTANCE DECREASING MAPS

A projective mapping $f: M \to M'$ betwen two Riemannian manifolds is distance decreasing provided M is complete and Ricci curvatures of M and M' are respectively bounded below and above[15,16,30].

In the case of space-time the concept of completeness is not quite the same, as M may be null-like and space-like complete without being time-like complete.

Since the time-like curves are carriers of particle-history, we consider manifolds with time-like completeness and prove a result similar to the Riemannian case.

More specifically we prove :

Theorem 6.1—Let $f: (M, g) \to (M', g')$ be a proper projective map between space-time manifolds M and M'. Assume that (M, g) is globally hyperbolic and time-like complete. If the Ricci curvature of M is bounded below by a constant $-k (k>0)$ and that of M' is bounded above by $k' > 0$, then f is distance decreasing up to a constant $(k/k')^{1/2}$ in the following sense : If X is a unit time-like vector field on M and

$X' = f_* X$ on M', then $- < f_* X, f_* X >' \leqslant \left(\frac{k}{k'}\right)$. The proof is based on Omari-Yau's maximum principle, whose version for Riemannian manifolds reads as follows[29*].

Lemma —Let M be a complete Riemannian manifold with Ricci curvature bounded below and let v be a c^2-function on M. Then for any $\alpha > 0$ and $\delta > - \sup v$, there exists a sequence (p_ν) in M such that

$$\text{(i)} \quad \lim_{\nu \to \infty} v(p_\nu) = \sup v$$

$$\text{(ii)} \quad \lim_{\nu \to \infty} \frac{\|dv(p_\nu)\|}{|v(p_\nu) + \delta|^{1+\alpha}} = 0$$

$$\text{(iii)} \quad \lim \frac{((\Delta v)(p_\nu))}{|v p_\nu) + \delta|^{1+2\alpha}} \leqslant 0.$$

To prove the theorem we establish a minor result (Proposition 6.2) by treating projective mapping simply as a reparametrisation of geodesics in (M, g).

We consider a time-like geodesic $\gamma = \gamma(t)$ in M and denote by $\bar{\gamma}$ its reparametrisation given by the mapping $\phi : I \to \bar{I}$. Evidently :

$$\bar{\gamma} = \gamma \circ \phi^{-1}. \qquad \ldots(6.2)$$

In the classical sense of projectively relatendess γ and $\bar{\gamma}$ are two geodesics associated with projectively related connections ∇ and $\bar{\nabla}$ [see eqn. (0.1)].

The differential equality

$$\frac{\cdot}{\gamma} = \frac{\dot{\gamma}}{D\phi} \circ \phi^{-1}. \text{≠} \qquad \ldots(6.3)$$

leads to

$$\bar{\nabla} \frac{\cdot}{D\gamma} = \left[\frac{\bar{\nabla} D\dot{\gamma}}{D\phi} - \frac{(D^2\phi)\,\dot{\gamma}}{(D\phi)^2}\right] \circ \phi^{-1} = 0$$

or equivalently to

$$\bar{\nabla}_{D\dot{\gamma}} = \frac{D^2\phi}{D\phi} \frac{\cdot}{\gamma}. \qquad \ldots(6.4)$$

*A related result for Lorentzian manifolds is proved in Prakash[18].

≠ '*' denotes the derivative w. r. to affine parameter and $D \equiv \frac{}{dt}$.

The relation (0.1) in terms of ∇, $\bar{\nabla}$ reads as

$$(\bar{\nabla}_x - \nabla_x)\, y = \omega(x)\, y + \omega(y)\, x \qquad \ldots(6.5)$$

where ω is a 1-form on M and x and y are arbitrary vectorfields on M. Corresponding curvature tensors satisfy :

$$\bar{R}(x, y)\, z = R(x, y)\, z + d\omega(x, y)\, z + (\Box\omega)(x,z)\, y - (\Box\omega)(y, z)\, x. \qquad \ldots(6.6)$$

The symbol $(\Box\omega)(x, y)$ stands for:

$$(\Box\omega)(x, y) = (\nabla_x\omega)(y) - \omega(x)\,\omega(y) \qquad \ldots(6.7)$$

and $d\omega$ is the usual exterior derivative of 1-form defined by :

$$2d\omega(x, y) = (\nabla_x\omega)\, y - (\nabla_y\omega)\, x - \omega[x, y]. \qquad \ldots(6.8)$$

In terms of $\mathrm{Ric}(y, z) = \mathrm{tr}(x \to R(x, y)z)$ the above eqn. (6.6) becomes :

$$\mathrm{Ric}(y, z) = \mathrm{Ric}(y, z) + d\omega(y, z) - 3(\Box\omega)(y, z). \qquad \ldots(6.9)$$

From (6.4) and (6.5), we can write (6.7) as

$$(\Box\omega)(\dot{\gamma}, \dot{\gamma}) = \frac{1}{2}\left[D\left(\frac{D^2\phi}{D\phi}\right) - \frac{1}{2}\left(\frac{D^2\phi}{D\phi}\right)^2\right]. \qquad \ldots(6.10)$$

The expression within braces on R. H. S. of (6.10) is the Schwartzian differential equation involving ϕ, accordingly, we have :

$$(\Box\omega)\ (\dot{\gamma}, \dot{\gamma}) = \tfrac{1}{2}\{\phi, t\} = \tfrac{1}{2} S_\phi. \qquad \ldots(6.11)$$

Hence we have shown :

Proposition 6.2—Let $\phi : I \to \bar{I}$ define a reparamatrisation between two projectively related geodesics then :

$$\bar{\mathrm{Ric}}(\dot{\bar{\gamma}}, \dot{\bar{\gamma}}) = \mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) - \tfrac{3}{2}\, S_\phi. \qquad \ldots(6.12)$$

We now use the projective parameters p, $\bar{p}$, and, in view of (3. 36), have :

$$S_p = \tfrac{3}{2}\,\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \qquad \ldots(6.13)$$

•Equation (6.6) is the coordinate-free from of eqn. (3.29).

$$S_{\bar{p}}^{-} = \tfrac{2}{3}\,\mathrm{Ric}\,(\dot{\bar{\gamma}},\dot{\bar{\gamma}}) = \tfrac{2}{3}\left[\mathrm{Ric}\left(\frac{\dot{\gamma}}{D\phi'},\ \frac{\dot{\gamma}}{D\phi}\right)\right]\circ\phi^{-1}$$

$$= \tfrac{2}{3}\frac{1}{(D\phi)^2}\,[\bar{\mathrm{Ric}}\,(\dot{\gamma},\dot{\gamma})\circ\phi^{-1}. \qquad \ldots(6.14)$$

From the chain rule for $\bar{p}\circ\phi$ we have

$$S^{-}_{\bar{p}\circ\phi} = (D\phi)^2\,(S^{-}_{\bar{p}})\circ\phi + S_{\phi} \qquad \ldots(6.15)$$

therefore using (6.14) and (6.12), we get

$$S_{p\circ\phi} = \tfrac{2}{3}\,\mathrm{Ric}\,(\dot{\gamma},\ \dot{\gamma}) = S_p. \qquad \ldots(6.16)$$

Equations (6.16) and (6.14) imply that if $\bar{p}$ is a projective parameter for the $\bar{\nabla}$ geodesic $\bar{\gamma}$ then $\bar{p}\circ\phi$ is the projective parameter for the ∇-geodesic $\gamma = \bar{\gamma}\circ\phi$. The equality $S^{-}_{\bar{p}\circ\phi} = S_p$ also says that p and $\bar{p}$ satisfy the conditions for a prope rprojective map.

Hence returning to the general situation $f:(M,g)\to(M',g')$ being a proper projective map we have the following relation between the Ricci tensors of M and M':

$$(f^{*}\ \bar{\mathrm{Ric}})\,(\dot{\gamma},\dot{\gamma}) = \mathrm{Ric}\ (\dot{\gamma},\dot{\gamma}) - \tfrac{3}{2}\,S_{\phi}{}^{*} \qquad \ldots(6.17)$$

and the proof for our theorem is as follows :

PROOF : Let X be a unit time-like vector-field on M : we shall show that $0 < -<f_{*}X, f_{*}X>' = -\|f_{*}X\|^2 \leqslant k/k'$.

Let γ be the unit-speed time-like geodesic in M such that $\dot{\gamma}(0) = X$. Set

$$u = <f_{*}X, f_{*}X>'\ (u<0).$$

Then

$$\nabla'_{D}<f_{*}X, f_{*}X>' = 2<\nabla'_{D}f_{*}X, f_{*}X>' = Du. \qquad \ldots(6.18)$$

But $f^{*}\dot{\gamma}$ denotes the tangent vector of time-like geodesic $\bar{\gamma} = \gamma\circ\phi$ in M', therefore,

$$\frac{D^2\phi}{D\phi} = \frac{Du}{2u}$$

---

*f^{*} is the map dual to the linear map $f_{*}: TM\to TM'$ induced by f, and $\bar{\mathrm{Ric}}$ is the Ricci tensor for M'.

and

$$S_\phi = \frac{1}{2}\frac{D^2u}{u} - \frac{5}{8}\left(\frac{Du}{u}\right)^2. \qquad \ldots(6.19)$$

Thus

$$\bar{\mathrm{Ric}}\,(f_*\dot{\gamma}, f_*\dot{\gamma}) = \mathrm{Ric}\,(\dot{\gamma}, \dot{\gamma}) - \frac{3}{4}\frac{D^2u}{u} + \frac{15}{16}\left(\frac{Du}{u}\right)^2.$$

Now $\mathrm{Ric}\,(\dot{\gamma}.\ \dot{\gamma}) \geqslant -k\|\dot{\gamma}\|^2 \geqslant -k$ and $\bar{\mathrm{Ric}}\,(f_*\ \dot{\gamma}, f_*\ \dot{\gamma}) \leqslant k'\|f_*\ \dot{\gamma}\|^2 = k'\,u$. Hence we have

$$k'\,u \geqslant -k - \frac{3}{4}\frac{D^2u}{u} + \frac{15}{16}\left(\frac{Du}{u}\right)^2$$

or

$$k'\,u \geqslant -k - \frac{3}{4}\frac{D^2u}{u}.$$

Writing u as $-v$ $(v > 0)$ we have

$$-k'\,v \geqslant -k - \frac{3}{4}\frac{D^2v}{v}$$

or

$$k'\,v \leqslant k + \frac{3}{4}\frac{D^2v}{v}$$

or

$$v\left(v - \frac{k}{k'}\right) \leqslant \frac{3}{4}\,D^2\,v\,.$$

Since M is time-like complete, we can choose a sequence of real numbers $\{p_\nu\}$ with the property $\lim_{\nu\to\infty} v\,(p_\nu) = \sup\ v\,(< \infty)$ and

$$\lim \frac{D^2v\ (p_\nu)}{(v\,(p_\nu) + \delta)^{1+2\alpha}} \leqslant 0.$$

Using the conclusions of Lemma for $\alpha < \frac{1}{2}$ we obtain the result :

$$0 < \sup v \leqslant k/k'.$$

7. Simple Examples of Projective Maps in Space-Time

Example 7.1—Let $f : (M, g) \to (M', g')$ be a diffeomorphism which carries the Lorentzian distance function on M to Lorentzian distance function on M', and M be globally hyperbolic then f is a projective map. Recall that for $p \in M$ and $q \in J^+(p)$ (the causal future of p) the Lorentzian distance function is defined as $d(p, q) = \sup \{L_g(\gamma) : \gamma \in \Omega_{p,q}\}$ and $d(p, q) = 0$ if $q \notin J^+(p)$, where

$$L_g(\gamma) = \sum_{i=0}^{n-1} \int_{t=t_i}^{t_{i+1}} \sqrt{-g(\gamma'(t), \gamma'(t))}\, dt \qquad \ldots(7.1)$$

and $\Omega_{p,q}$ is the path-space of all future directed non-space like curves joining p to q.

Since (M, g) is globally hyperbolic the distance function is finite and continuous, moreover the first variation of distance function joining p and q gives the geodesic of maximal length between these points Therefore, the distance function carried by diffeomorphism f to (M', g') is finite and continuous whose first variation gives the maximal geodesic joining $f(p)$ and $f(q)$. Moreover f satisfies (6.17) and thus preserves the projective parameter as well.

Example 7.2 – Consider giving up the global hyperbolicity condition on M, then f restricted to time-like curves is a projective map. Extremals of first-variation of (7.1) are geodesics on M, since $L_g(\gamma) = L_{g'}(f \circ \gamma)$, and (6.17) is satisfied the result is obvious.

Recall that a warped product $M \times_f H = (\bar{M}, \bar{g})$ is a Lorentzian manifold which is defined as follows :

M is Lorentzian, $f : M \to (0, \infty)$ is a smooth function, and H is an arbitrary Riemann manifold. The metric $\bar{g}$ is given by the equality:

$$\bar{g}(\bar{v}, \bar{w}) = g(\pi_* \bar{v}, \pi_* \bar{w}) + f(\pi(\bar{p}))\, h(\eta_* \bar{v}, \eta_* \bar{w}) \qquad \ldots(7.2)$$

where $\bar{v}, \bar{w} \in T_p(\bar{M})$ and $\pi : M \times_f H \to M$, $\eta : M \times_f H \to H$ are the first and second projections of the warped product. The familiar example of warped product is the exterior Schwartzschild space-time $(M \times_f H, \bar{g})$ where $\bar{g} = g \oplus fh$, h being the Riemannian metric of 2-sphere $H = S^2$, induced by the imbedding $S^2 \to \mathbf{R}^3$, g the Lorentzian metric :

$$g = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2$$

for $M = \{(t, \gamma) \in \mathbf{R}^2 : \gamma > 2m\}$ and $f : M \to \mathbf{R}$ being the function $f(t, r) = r^2$. The example we have in mind is : M one-dimensional with negative definite metric and H a 3-dimensional Riemannian manifold.

Example 7.3—Let $(\bar{M}_1, \bar{g}_1) = (-(dt)^2 \times_f H_1)$ and $(\bar{M}_2, \bar{g}_2) = (-(dt)^2 \times_f H_2)$ be two space time manifolds such that ϕ: $(H_1, g_1) \to (H_2, g_2)$ is projective, then it is easy to check that $(\bar{M}_1, \bar{g}_1)$ and $(\bar{M}_2, \bar{g}_2)$ are related to each other by a projective map Φ which is obviously an identity map on the first factor of this product and coincides with ϕ on the other factor.

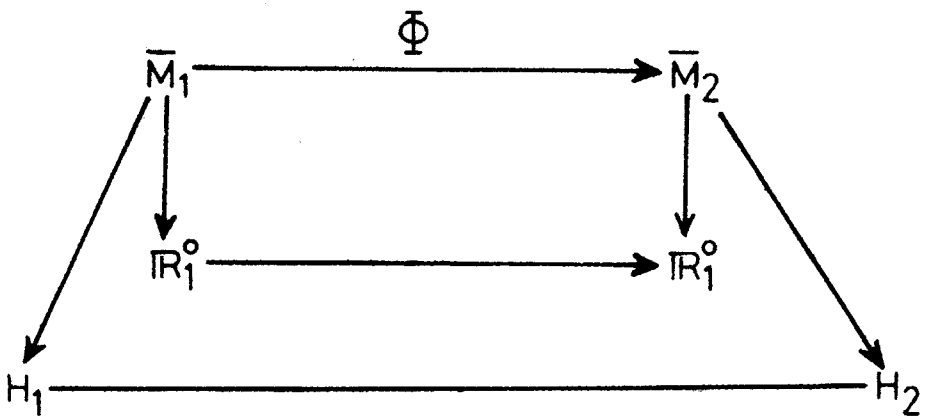

FIG. 3.

From the above example it is obvious that $f = f(t)$ which we have taken to be the same for both $\bar{M}_1$ and $\bar{M}_2$ can be taken to be different, provided we assume that $\Phi^* \circ f_2 = f_1$ for every t in R.

Example 7.4—Let H_1 and H_2 be respectively the unit sphere S^3 and the projective space $\mathbb{P}^3(\mathbf{R})$ the warped products $\bar{M}_1$ and $\bar{M}_2$ for some f are trivially, projectively related.

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APPENDIX 1

To prove the equality :

$$\{p, s\} = -\{s, p\} \Big/ \left(\frac{ds}{dp}\right)^2$$

we observe that :

$$\frac{dp}{ds} = \left(\frac{ds}{dp}\right)^{-1} \qquad \ldots(1)$$

implies :

$$\frac{d^2p}{ds^2} = \frac{d}{dp}\left(\left(\frac{ds}{dp}\right)^{-1}\right)\Big/\left(\frac{ds}{dp}\right)$$

$$= -\frac{d^2s}{dp^2}\Big/\left(\frac{ds}{dp}\right)^3. \qquad \ldots(2)$$

Similarly

$$\frac{d^3p}{ds^3} = -\frac{d}{dp}\left(\left(\frac{d^2s}{dp^2}\right)\Big/\left(\frac{ds}{dp}\right)^3\right)\Big/\left(\frac{ds}{dp}\right)$$

$$= -\frac{d^3s}{dp^3}\Big/\left(\frac{ds}{dp}\right)^4 + 3\left(\frac{d^2s}{dp^2}\right)^2\left(\frac{ds}{dp}\right)^5. \qquad \ldots(3)$$

Accordingly, we have :

$$\{p, s\} = \left(\frac{d^3p}{ds^3}\right)\Big/\left(\frac{dp}{ds}\right) - \frac{3}{2}\left(\frac{d^2p}{ds^2}\right)^2\Big/\left(\frac{dp}{ds}\right)^2$$

$$= -\left(\frac{d^3s}{dp^3}\right)\Big/\left(\frac{ds}{dp}\right)^3 + 3\left(\frac{d^2s}{dp^2}\right)^2\Big/\left(\frac{ds}{dp}\right)^4$$

$$-\frac{3}{2}\left(-\frac{d^2s}{dp^2}\right)^2\Big/\left(\frac{ds}{dp}\right)^4 = -\left[\left(\frac{d^3s}{dp^3}\right)\Big/\left(\frac{ds}{dp}\right)\right.$$

$$\left.-\frac{3}{2}\left(\frac{d^2s}{dp^2}\right)^2\Big/\left(\frac{ds}{dp}\right)^2\right]\Big/\left(\frac{ds}{dp}\right)^2 = -\{s, p\}\Big/\left(\frac{ds}{dp}\right)^2$$

which is (3.10).

Appendix 2

Third order differential equation

$$\{p, t\} \equiv \frac{d^3p}{dt^2}\Big/\frac{dp}{dt} - \frac{3}{2}\left(\frac{d^2p}{dt^2}\Big/\frac{dp}{dt}\right)^2 = \pm\eta^2 \qquad \ldots(1)$$

can be reduced to first order differential equation by setting

$$\frac{d^2p}{dt^2}\Big/\frac{dp}{dt} = f. \qquad \ldots(2)$$

Thus (1) becomes

$$\frac{df}{dt} - \frac{1}{2} f^2 = \pm \eta^2. \qquad \ldots(3)$$

Using the + sign

$$\frac{df}{f^2 + 2\eta^2} = \frac{1}{2} dt \qquad \ldots(4)$$

gives

$$\frac{d^2p}{dt^2} \Big/ \frac{dp}{dt} = \sqrt{2\eta} \tan\left(\frac{\sqrt{2\eta}}{2} t + \sqrt{2\eta}\alpha'\right), \text{ or} \qquad \ldots(5)$$

$$\ln \left| \frac{dp}{dt} \right| = 2 \ln | \sec \left(\frac{\sqrt{2\eta}}{2} t + \sqrt{2\eta}\alpha'\right) | + \ln \beta', \qquad \ldots(6)$$

or

$$\frac{dp}{dt} = \beta' \sec^2\left(\frac{\sqrt{2\eta}}{2} t + \sqrt{2\eta}\alpha'\right), \qquad \ldots(7)$$

or

$$p = \sqrt{2}\frac{\beta'}{\eta} \tan\left(\frac{\sqrt{2\eta}}{2} t + \sqrt{2\eta}\alpha'\right) + \gamma. \qquad \ldots(8)$$

Adjusting the constants :

$$p = \beta \tan \{(\sqrt{2\eta}t + \alpha)/2\} + \gamma. \qquad \ldots(9)$$

This is (4.2) i'.

Similarly using the−sign, we can write (3) as :

$$\frac{1}{\sqrt{2\eta}} \int \left(\frac{1}{f - \sqrt{2\eta}} - \frac{1}{f + \sqrt{2\eta}} \right) df = \int dt, \qquad \ldots(10)$$

or

$$\ln \left| \frac{f - \sqrt{2}_N}{f + \sqrt{2\eta}} \right| = \sqrt{2\eta}\ (t + \alpha'), \qquad \ldots(11)$$

or

$$f = \sqrt{2\eta}\ [1 + \exp \sqrt{2\eta}\ (t + \alpha')/1 - e\sqrt{2\eta}\ (t + \alpha')] \qquad \ldots(12)$$

$$\frac{d^2p}{dt^2} \Big/ \frac{dp}{dt} = 2\eta \coth \frac{u}{2} \qquad \ldots(13)$$

where we have denoted $\sqrt{2\eta}\ (t + \alpha')$ as u.

$$\ln \left| \frac{dp}{dt} \right| = 2 \ln | \text{Cosch } \frac{u}{2} | + \ln \beta', \qquad \ldots(14)$$

or

$$\frac{dp}{dt} = \beta' \text{ Csch}^2 \frac{u}{2} \text{ or rewriting } u \text{ in terms of } t \qquad \ldots(15)$$

$$p = \frac{-2\beta'}{\sqrt{2\eta}} \text{ Coth } (\sqrt{2\eta}t + \sqrt{2\eta}\alpha')/2 + \gamma'$$

$$= \frac{\sqrt{2}}{\eta}\beta' \left(\frac{2}{1 - \exp(\sqrt{2\eta}\,(t + \alpha'))} - 1 \right) + \gamma'. \qquad \ldots(16)$$

Adjusting the constants, we have (4.2) *ii'*.

When $\eta = 0$ the differential equation $\{p, t\} = 0$ can again be solved using the equation $\frac{df}{dt} - \frac{1}{2} f^2 = 0$ This gives :

$$\frac{d^2p}{dt^2} \Big/ \frac{dp}{dt} = -\frac{2}{t + \alpha}, \text{ or} \qquad \ldots(17)$$

$$\ln \left| \frac{dp}{dt} \right| = -2 \ln (t + \alpha) + \ln \beta', \text{ or} \qquad \ldots(18)$$

$$p = -\frac{\beta'}{t + \alpha} + \gamma \qquad \ldots(19)$$

or writing β in place of $-\beta'$ we have

$$p = \frac{\beta}{t + \alpha} + \gamma \qquad \ldots(20)$$

which is (4.2) *iii'*.

Appendix 3

We write (3.9) in terms of $(\bar{x}^i)$ and equate the R. H. S. of both these to obtain the relation

$$\Gamma^4_{jk} = \Gamma^4_{l\cdot m} \frac{\partial x^l}{\partial \bar{x}^j} \frac{\partial x^m}{\partial \bar{x}^k} \qquad \ldots(1)$$

which shows that Γ^{4}_{jk} are components of covariants tensor of order two, even though notationally they appear coefficients of connection.

Also since parameter p is an invariant of projective change of affine connection, we can write (3.9} in terms of another affine parameter $\hat{s}$, thus

$$\{p, \hat{s}\} = -2\lambda \hat{\Gamma}^{4}_{jk} \frac{dx^{j}}{d\hat{s}} \frac{dx^{k}}{d\hat{s}} . \qquad \ldots(2)$$

We use the identity

$$\{p, \hat{s}\} = \left(\frac{ds}{d\hat{s}} \right)^{2} \left[\{p, s\} - \{\hat{s}, s\} \right] \qquad \ldots(3)$$

and (3.5) to obtain

$$\hat{\Gamma}^{4}_{jk} = \Gamma^{4}_{jk} + \frac{1}{\lambda} \left\{ \frac{1}{2} \left(\frac{\partial p_{j}}{\partial x^{k}} + \frac{\partial p_{k}}{\partial x^{j}} \right) - p_{r} \Gamma^{r}_{jk} + p_{j} p_{k} \right\}. \qquad \ldots(4)$$

Above equation is the transformation equation for the tensor Γ^{4}_{jk} under the projective of affine connection Γ.