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ON EXTENSIONS OF σ-ADDITIVE SET FUNCTIONS WITH VALUES IN A TOPOLOGICAL GROUP

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Two types of extensions of a topological group (G)-valued σ-additive set functions defined on a σ-algebra $\mathcal{M}$ of subsets of a set X have been discussed. Firstly, extension of a σ-quasi-measure $\mu: \mathcal{M} \to G$ to a σ-quasi-measure $\nu: \mathcal{M}^{\tau}_{\mu} \to G$ has been considered. Next, a σ-quasi-measure $\mu: \mathcal{M} \to G$ has been extended to a σ-quasi-measure $\nu: [\mathcal{M}, Z] \to G$, where $Z \subset X$ and $Z \notin \mathcal{M}$.

1. INTRODUCTION

This paper discusses the extensions of some results of Lipecki[2]. He has studied two types of extensions of an additive set function $\mu: \mathcal{M} \to G$, where $\mathcal{M}$ is an algebra of sets and G is an Abelian group. With discrete topology δ on G he has obtained an extension of μ to an algebra $\mathcal{M}^{\delta}_{\mu}$ containing $\mathcal{M}$ from which μ can not be extended further. In fact, this type of extension has been obtained in case G is a topological semi-group with zero. Also, taking G to be a complete Hausdorff topological group, he has extended a quasi-measure μ to an algebra $\mathcal{M}^{\tau}_{\mu}$ containing $\mathcal{M}$, τ being the topology on G which admits no further extension. Finally, he has proved that a quasi-measure $\mu: \mathcal{M} \to G$, where G is a complete Hausdorff topological group, can be extended to a quasi-measure ν on the power set 2^X so as to ensure denseness of $\mathcal{M}$ in 2^X with respect to the topology generated by ν on 2^X.

We have proved in the first place that if we take μ to be σ-additive and $\mathcal{M}$ to be a σ-algebra, then μ admits of an extension to a σ-additive set function ν on a σ-algebra $\mathcal{M}^{\delta}_{\mu}$ containing $\mathcal{M}$, where δ is the discrete topology on G. This set function ν admits no further extension. However, this type of extension has been obtained

even when G is a topological semi-group with zero. Further taking G to be ac omplete Hausdorff topological group, a σ-quasi-measure μ has been extended to a σ-quasi-measure ν on a σ-algebra $\mathcal{M}_\mu^\tau$ containing $\mathcal{M}$, where τ is the topology on G, using net limits μ^* and μ_*. The process of extension of μ is complete in the sense that no further extension of ν to a σ-algebra cantaining $\mathcal{M}_\mu^\tau$ is possible.

In the second place we have been able to obtain an extension of a σ-quasi-measure $\mu : \mathcal{M} \to G$, where G is a complete Hausdorff topological group, to a σ-quasi-measure ν on $[\mathcal{M}, Z]$, the σ-algebra generated by $\mathcal{M}$ and an additional set $Z \subset X$ such that $\mathcal{M}$ is dense in $[\mathcal{M}, Z]$ in the ν-topology. The quesion of its being extended to 2^X still remains open.

2. Extension to $\mathcal{M}_\mu^\tau$

Let $\mathcal{M}$ be a σ-algebra of subsets of a set X and let $(G, +, 0, \tau)$ by an Abelian topological semi-group with zero. Throughout this paper the letters U, V and W sometimes accompanied by indices stand for neighbourhoods of O in the topology τ in G.

Let $\mu : \mathcal{M} \to G$ be a set function. We denote by $\mathcal{M}_\mu^\tau$ (or briefly by $\mathcal{M}_\mu$ when τ is fixed) the family of all $E \subset X$ for which to every V there exist $M, N \in \mathcal{M}$ such that

$$M \subset E \subset N \text{ and } \mu(S) \in V \text{ is } N \setminus M \supset S \in \mathcal{M}. \qquad \ldots(1)$$

Theorem 2.1—If $\mu : \mathcal{M} \to G$ is a σ-additive set function with $\mu(\phi) = 0$ and τ is regular, then $\mathcal{M}_\mu^\tau$ is a σ-algebra of subsets of X containing $\mathcal{M}$.

Proof : Suppose $E \in \mathcal{M}_\mu^\tau$. Then to every V there exist $M, N \in \mathcal{M}$ such that $M \subset E \subset N$ and $\mu(S) \in V$ if $N \setminus M \supset S \in \mathcal{M}$. Clearly $N^c \subset E^c \subset M^c$ and $N \setminus M = M^c \setminus N^c$. Thus to every V and E^c there exist $M^c, N^c \in \mathcal{M}$ such that $N^c \subset E^c \subset M^c$ and $\mu(S) \in V$ if $M^c \setminus N^c \supset S \in \mathcal{M}$. Hence $E^c \in \mathcal{M}_\mu^\tau$.

Next, let $E_i \in \mathcal{M}_\mu^\tau$, $i = 1, 2, \ldots$

Let U be chosen arbitrarily. Then there exists V such that $\bar{V} \subset U$. Also, there exist $V_1, V_2, \ldots, V_i$ such that $V_1 + V_2 + \ldots + V_i \subset V$, $i = 1, 2, \ldots$. Corresponding to E_i and V_i there exist $M_i, N_i \in \mathcal{M}$ such that $M_i \subset E_i \subset N_i$ and $\mu(S) \in V_i$ if $N_i \setminus M_i \supset S \in \mathcal{M}$, $i = 1, 2, \ldots$.

Now,

$$\bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} N_i.$$

Let

$$S \subset \bigcup_{i=1}^{\infty} N_i \setminus \bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} (N_i \setminus M_i), \text{ where } S \in \mathscr{M}.$$

We can now write

$$S = \bigcup_{i=1}^{\infty} S_i' \quad , \text{ where } S_i' = S \cap (N_i \setminus M_i),\ S_i' \in \mathscr{M}$$

$$= \bigcup_{i=1}^{\infty} S_i'' \quad , \text{ where } \left\{ S_i'' \right\} \text{ is a mutually disjoint sequence}$$

and

$$S_i'' \subset S_i' \quad , \quad S_i'' \in \mathscr{M}.$$

Then

$$\mu(S) = \mu \left(\bigcup_{i=1}^{\infty} S_i'' \right)$$

$$= \sum_{i=1}^{\infty} \mu \left(S_i'' \right) = \text{since } \mu \text{ is } \sigma\text{-additive}$$

$$= \lim_n \left\{ \sum_{i=1}^{n} \mu \left(S_i'' \right) \right\}$$

$$= \lim_n \left\{ \mu \left(\bigcup_{i=1}^{n} S_i'' \right) \right\}$$

$$= \lim_n \left\{ \mu \left(P_n'' \right) \right\} \in \bar{V} \subset U$$

where

$$P_n'' = S_1'' \cup S_2'' \cup \ldots \cup S_n''$$

since

$$\mu\left(P_n'' \right) \in V_1 + V_2 + \ldots + V_n \subset V.$$

It follows that

$$\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}_\mu^\tau .$$

Also, since $\mu(\phi) = 0$, it follows that $\mathcal{M} \subset \mathcal{M}_\mu^\tau$.

This completes the proof.

In the following discussion δ will denote the discrete topology on G.

It is easy to see that for $E \in \mathcal{M}_\mu^\delta$, there exist $M, N \in \mathcal{M}$ such that

$$\mu(S) = 0 \qquad \ldots(1')$$

whenever $M \subset E \subset N$ and $N \setminus M \supset S \in \mathcal{M}$ and that $\mathcal{M}_\mu^\delta \subset \mathcal{M}_\mu^\tau$ for any topology τ on G.

Theorem 2.2—Let $\mu : \mathcal{M} \to G$ be σ-additive and $\mu(\phi)=0$. Then the set function $\nu : \mathcal{M}_\mu^\delta \to G$, defined by $\nu(E) = \mu(M)$, where $M, N \in \mathcal{M}$ and E satisfies (1'), is σ-additive.

Moreover,

$$\nu \mid \mathcal{M} = \mu \text{ and } \left(\mathcal{M}_\mu^\delta \right)_\nu^\delta = \mathcal{M}_\mu^\delta .$$

PROOF : As in Theorem 1 of Lipecki[2] we can show that ν is well-defined on $\mathcal{M}_\mu^\delta$

Let $E_i \in \mathcal{M}_\mu^\delta$, $i = 1, 2, \ldots$, and $E_i \cap E_j = \phi$, $i \neq j$.

We take

$M_i, N_i \in \mathcal{M}$ as in (1').

We then have

$$\bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} N_i.$$

Let

$$S \in \mathscr{M} \text{ and } S \subset \bigcup_{i=1}^{\infty} N_i \setminus \bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} (N_i \setminus M_i).$$

Since μ is σ-additive, it can be shown that

$$\mu(S) = \mu\{S \cap (N_1 \setminus M_1)\} + \mu\{S \setminus (N_1 \setminus M_1)\} = 0.$$

Also,

$$\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} M_i\right)$$

$$= \sum_{i=1}^{\infty} \mu(M_i)$$

$$= \sum_{i=1}^{\infty} \nu(E_i).$$

Thus ν is σ-additive.

It is easy to see that $\nu \mid \mathscr{M} = \mu$.

Also, following the arguments of Lipecki[2] (Theorem 1) we can easily show that $\left(\mathscr{M}_{\mu}^{s}\right)_{\mu}^{s} = \mathscr{M}_{\mu}^{s}$.

Theorem 2.3—Let τ be a topology on G such that there is a countable family $\{V_i\}$ of neighbourhoods of zero with $\bigcap_{i=1}^{\infty} V_i = \{0\}$. Then for any set function $\mu : \mathscr{M} \to G$, $\mathscr{M}_{\mu}^{\tau} = \mathscr{M}_{\mu}^{s}$.

Proof is exactly same as that of Proposition 2 of Lipecki[2].

Definition 2.1—A σ-additive set function $\mu : \mathscr{M} \to G$ is called a σ-quasi-measure if it is exhaustive. i. e., for any sequence $\{M_i\}$ of pairwise disjoint sets in $\mathscr{M}$, $\mu(M_i) \to 0$.

Lemma 2.1—Let $\mu : \mathcal{M} \to G$ be a σ-quasi-measure. To every V and $E \subset X$, there exist $M, N \in \mathcal{M}$ such that

$$M \subset E \text{ and } \mu(M \setminus M) \in V \text{ if } M \subset M' \subset E,\ M' \in \mathcal{M}, \tag{2}$$

$$N \supset E \text{ and } \mu(N \setminus N') \in V \text{ if } E \subset N' \subset N,\ N' \in \mathcal{M}. \tag{3}$$

Proof is exactly same as that of Lemma 1 of Lipecki[2].

For the subsequent discussion we assume that $(G, +, \tau)$ is an Abelian, Hausdorff topological group. Since the topology τ is fixed, we shall write $\mathcal{M}_\mu$ in place of $\mathcal{M}^\tau_\mu$.

Lemma 2.2—Let $\mu : \mathcal{M} \to G$ be a σ-additive set function.

The family $\mathcal{M}_\mu$ consists of all sets $E \subset X$ for which to every V there exist $M, N \in \mathcal{M}$ such that

$$M \subset E \subset N \text{ and } \mu(S') - \mu(S'') \in V \tag{1''}$$

if

$$M \subset S', S'' \subset N$$

and

$$S', S'' \in \mathcal{M}.$$

Proof is exactly same as that of Remark 3 of Lipecki[2].

Let us now assume in addition that G is complete. Let $\mu : \mathcal{M} \to G$ be a σ-quasi-measure.

For any $E \subset X$ we put

$$\mu_*(E) = \lim \{\mu(M) : E \supset M \in \mathcal{M}\} \tag{4}$$

and

$$\mu^*(E) = \lim \{\mu(N) : E \subset N \in \mathcal{M}\}, \tag{5}$$

where the sets of indices are directed by inclusion and its converse respectively. Since G is complete and Hausdorff, it follows from Lemma 2.1 that both the limits exist and are unique.

Lemma 2.3—The set functions μ_* and μ^* defined by (4) and (5) respectively, have the following properties.

(i) $\mu(M) = \mu_*(M) = \mu^*(M)$ for $M \in \mathscr{M}$;

(ii) $\mu_*(E) = \mu^*(E)$ for $E \in \mathscr{M}_\mu$;

(iii) $\mu_*(E_1 \cup E_2) = \mu_*(E_1) + \mu_*(E_2)$, $E_1 \cap E_2 = \phi$, $E_1, E_2 \in \mathscr{M}_\mu$;

(iv) $\mu_*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu_*(E_i)$, where $\{E_i\}$ is a mutually disjoint sequence of $\mathscr{M}_\mu$.

PROOF : (i) : It is obvious.

(ii) Let $E \in \mathscr{M}_\mu$ and W be chosen arbitrarily. Then there exists V such that $V + V - V \subset W$.

Using the definitions of μ, μ^* and Lemma 2.2 we can show that

$$\mu_*(E) - \mu^*(E)_* \in V + V - V \subset W.$$

Since W is arbitrary and $\cap W = 0$, it follows that $\mu_*(E) = \mu^*(E)$.

(iii) : The proof is exactly same as that of Lemma 2 (iii) of Lepecki[2].

(iv) : Since μ_* is additive by (iii), it suffices to show that $\mu_*(F_i) \to 0$, where $\{F_i\} \subset X$ such that $F_i \downarrow \phi$.

Let W be chosen arbitarily. We take V such that $V + V \subset W$. Corresponding to F_i and V there exists $M_i \in \mathscr{M}$ such that $M_i \subset F_i$

and $\mu_*(F_i) \in \mu(M) + V$ for all $M \in \mathscr{M}$ satisfying $M_i \subset M \subset F_i$.

Let

$$P_i = \bigcup_{j=i}^{\infty} M_i \subset F_i, \text{ since } F_i \downarrow .$$

Clearly $P_1 \supset P_2 \supset \ldots$ and $\bigcap_{i=1}^{\infty} P_i = \phi$. Thus $P_i \downarrow \phi$.

Since

$$P_i \in \mathscr{M} \text{ and } M_i \subset P_i \subset F_i, \text{ it follows that } \mu_*(F_i) \in \mu(P_i) + V.$$

Again, since μ is σ-additive and exhaustive, and $\{P_i\} \subset \mathscr{M}$, $P_i \downarrow \phi$, by Lemma 3, of Lipecki[3] we can find a positive integer i_0 such that $\mu(P_i) \in V$ for $i \geq i_0$. Consequently, $\mu_*(F_i) \in V + V \subset W$ for $i \geq i_0$.

Thus $\mu_*(F_i) \to 0$. Hence μ_* is σ-additive on $\mathcal{M}_\mu$.

Definition 2.2—A σ-additive set function $\mu : \mathcal{M} \to G$ is called $\mathcal{K}$-tight ($\mathcal{K}$ being a sub-family of $\mathcal{M}$) if for every $M \in \mathcal{M}$ and V, there exists a $K \in \mathcal{K}$ such that $K \subset M$ and $\mu(M) - \mu(M') \in V$ whenever $K \subset M' \subset M$, $M' \in \mathcal{M}$.

Theorem 2.4—Let G be an Abelian, complete Hausdorff topological group. Every σ-quasi-measure $\mu : \mathcal{M} \to G$ has the unique extension to an $\mathcal{M}$-tight σ-quasi-measure $\nu : \mathcal{M}_\mu \to G$.

Moreover, $(\mathcal{M}_\mu)_\nu = \mathcal{M}_\mu$, $\nu_* = \mu_*$ and $\nu^* = \mu^*$.

PROOF : We put $\nu(E) = \mu_*(E)$ for $E \in \mathcal{M}_\mu$.

Let $M \in \mathcal{M}$. Then $\nu(M) = \mu_*(M) = \mu(M)$, by Lemma 2.3 (i).

Thus ν is an extension of μ.

Let

$$\{E_i\} \subset \mathcal{M}_\mu \text{ and } E_i \cap E_j = \phi,\ i \neq j,\ i, j = 1, 2, \ldots$$

Suppose

$$E = \bigcup_{i=1}^{\infty} F_i. \text{ Then } E \in \mathcal{M}_\mu.$$

Now,

$$\nu(E) = \mu_*\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$= \sum_{i=1}^{\infty} \mu_*(E_i), \text{ by Lemma 2.3 (iv)}$$

$$= \sum_{i=1}^{\infty} \nu(E_i).$$

Thus ν is σ-additive.

Using the definitions of μ_* and ν we can show that ν is $\mathcal{M}$-tight. The exhaustivity of ν follows easily from $\mathcal{M}$-tightness of ν and exhaustivity of μ. The uniqueness of ν is also an easy consequence of its $\mathcal{M}$-tightness.

We shall now prove that $(\mathcal{M}_\mu)_\nu = \mathcal{M}_\mu$.

Let W be chosen arbitrarily. We take V such that $V + V + V \subset W$. Let $F \in (\mathcal{SM}_\mu)_\nu$. Then by definition, there exist E_1, $E_2 \in \mathcal{SM}_\mu$ such that $E_1 \subset F \subset E_2$ and $\nu(S) \in V$ if $E_2 \setminus E_1 \supset S \neq \mathcal{SM}_\gamma$. In particular, $\mu(R) \in V$ for $E_1 \setminus E_1 \supset R \in \mathcal{SM}$.

We take M_i, $N_i \in \mathcal{SM}$ such that $M_i \subset E_i \subset N_i$, $i = 1, 2$ and $\mu(S) \in V$ if $N_i \setminus M_i \supset S \in \mathcal{SM}$, $i = 1, 2$. We have $M_i \subset F \subset N_i$ and $N_2 \setminus M_1 \subset (N_i \setminus M_i) \cup (E_2 \setminus E_1) \cup (N_2 \setminus M_2)$. Let $S \in \mathcal{SM}$ and $S \subset N_2 \setminus M_1$. It now follows by the additivity of μ that $\mu(S) \in W$. Thus $F \in \mathcal{SM}_\mu$.

Consequently, $(\mathcal{SM}_\mu)_\nu \subset \mathcal{SM}_\mu$.

Also, it is obvious that $\mathcal{SM}_\mu \subset (\mathcal{SM}_\mu)_\nu$.

Thus $(\mathcal{SM}_\mu)_\nu = \mathcal{SM}_\mu$.

Next, we shall show that $\nu_* = \mu_*$.

Let $E \subset X$. We have $\nu_*(E) = \lim \{\nu(M) : E \supset M \in \mathcal{SM}_\mu\}$

$= \lim \{\mu_*(M) : E \supset M \in \mathcal{SM}_\mu\}$.

Then there exists $M_0 \in \mathcal{SM}_\mu$ such that $\nu_*(E) - \mu_*(M) \in V$ for all $M \in \mathcal{SM}_\mu$ satisfying $M_0 \subset M \subset E$.

In particular, $\nu_*(E) - \mu(M) \in V$ for all $M \in \mathcal{SM}$ satisfying $\mathcal{SM}_0 \subset M \subset E$.

Let $M_0 \subset N_0 \subset E$, where $N_0 \in \mathcal{SM}$.

Then $\nu_*(E) - \mu(M) \in V$ for all $M \in \mathcal{SM}$ satisfying $\mathcal{SM}_0 \subset M \subset E$.

Thus $\nu_*(E) = \lim \{\mu(M): E \supset M \in \mathcal{SM}\}$

$= \mu_*(E)$.

Consequently, $\nu_* = \mu_*$.

Analogously we can prove that $\nu^* = \mu^*$.

This completes the proof.

Lemma 2.4— Let μ_* and μ^* be the same as in Lemma 2.3. Then

(v) $\mu_*(A) + \mu^*(B) = \mu_*(A \cup B)$ for $A, B \subset X$ such that $A \cup B \in \mathcal{SM}_\mu$ and $A \cap B = \phi$;

(vi) $\mu_*(A) + \mu_*(B) = \mu_*(A \cup B)$ for $A, B \subset X$ such that there exists an $E \in \mathcal{SM}_\mu$ with $A \subset E$ and $B \cap E = \phi$;

(vii) $\mu^*(A) + \mu^*(B) = \mu^*(A \cup B)$ for $A, B \subset X$ such that there exists an $E \in \mathcal{SM}_\mu$ with $A \subset E$ and $B \cap E = \phi$.

PROOF : In view of Theorem 2.4, it is enough to prove all the above statements with $\mathcal{M}_\mu$ replaced by $\mathcal{M}$. Proofs of (v) and (vi) are exactly same as those of Lemma 3 (iv), (v) of Lipecki[2]. For the proof of (vii) we choose arbitrarily and take V

such that $V - V - V \subset W$.

Using the definition of μ^* and the given condition we can show that

$$\mu^*(A \cup B) - \mu^*(A) - \mu^*(B) \in V - V - V \subset W.$$

It follows that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$.

Following the arguments of Lipecki[2], we can establish the following corollary:

Corollary—Let $E \subset X$ be an arbitrary set. The following conditions are equivalent:

(a) $E \in \mathcal{M}_\mu$;

(b) $\mu_*(Z) = \mu_*(Z \cap E) + \mu_*(Z \setminus E)$ for all $Z \subset X$;

(c) $\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \setminus E)$ for all $Z \subset X$;

(d) $\mu(S) = \mu_*(S \cap E) + \mu_*(S \setminus E)$ for all $S \in \mathcal{M}$;

(e) $\mu S) = \mu^*(S \cap E) + \mu^*(S \setminus E)$ for all $S \in \mathcal{M}$.

3. EXTENSION TO $[\mathcal{M}, Z]$

Let $\mathcal{M}$ be an algebra of subsets of X.

Then it can be considered as a group with the symmetric difference of sets Δ as group operation.

Let $\mu : \mathcal{M} \to G$ be an additive set function.

Then μ gives rise to a topology on $\mathcal{M}$ called the μ-topology whose neighbourhood base at ϕ is given by $\{M \in \mathcal{M}; \mu(S) \in V$ for all $M \supset S \in \mathcal{M}\}$, where V runs through some neighbourhood base at zero in G. (Proposition 1.9 of Drewnowski[1].)

Lemma 3.1—Let $\mathcal{M}$ and $\mathcal{N}$ be two σ-algebras of subsets of X and $\mathcal{M} \subseteq \mathcal{N}$. Let $\nu : \mathcal{N} \to G$ be σ-additive and $\mathcal{M}$ be a dense sub-σ-algebra in the ν-topology. If ν restricted to $\mathcal{M}$ is exhaustive, then it is also exhaustive on $\mathcal{N}$.
Proof is similar to that of Lemma 4 of Lepecki[2].

Let $\mathcal{M}$ and $\mathcal{N}$ be σ-algebras of subsets of X and let $\mu : \mathcal{M} \to G$ and $\nu : \mathcal{M} \to G$ be σ-additive. We write $(\mathcal{M}, \mu) \leqslant (\mathcal{M}, \nu)$ if and only if $\mathcal{M} \subset \mathcal{M}$, $\nu \mid \mathcal{M} = \mu$

and for any closed neighbourhood V of zero and $M \in \mathcal{M}$ such that $\mu(S) \in V$ whenever $M \supset S \in \mathcal{M}$, we have $\nu(N) \in V$ if $M \supset N \in \mathcal{M}$.

It is clear that the relation $\leqslant$ is a partial order and that if $(\mathcal{M}, \mu) \leqslant (\mathcal{M}, \nu)$, then the μ-topology coincides with the topology induced on $\mathcal{M}$ by the ν-topology.

If $\mathcal{M}$ be a family of subsets of X, $Z \subset X$, we denote by $[\mathcal{M}, Z]$ the family of all sets of the form $(M \cap Z) \cup (N \cap Z^c)$, where $M, N \in \mathcal{M}$.

Lemma 3.2—If $\mathcal{M}$ is a σ-algebra, then $[\mathcal{M}, Z]$ is the smallest σ-algebra containing $\mathcal{M}$ and Z.

Moreover, for any disjoint $E_1, E_2 \in [\mathcal{M}, Z]$, there exist disjoint M_1, M_2 and disjoint N_1, N_2 in $\mathcal{M}$ such that $E_i = (M_i \cap Z) \cup (N_i \cap Z^c)$, $i = 1, 2$.

PROOF : For the proof of the first part of the lemma we proceed as follows:

$[\mathcal{M}, Z]$ is an algebra[4] (p. 268).

Let

$$\{E_n\} \subset [\mathcal{M}, Z].$$

Now,

$$E_n = (M_n \cap Z) \cup (N_n \cap Z^c), \text{ where } M_n, N_n \in \mathcal{M}.$$

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} [(M_n \cap Z) \cup (N_n \cap Z^c)]$$

$$= [\bigcup_{n=1}^{\infty} (M_n \cap Z)] \cup [\bigcup_{n=1}^{\infty} (N_n \cap Z^c)]$$

$$= [(\bigcup_{n=1}^{\infty} M_n) \cap Z] \cup [(\bigcup_{n=1}^{\infty} N_n) \cap Z^c].$$

Thus

$$\bigcup_{n=1}^{\infty} E_n \in [\mathcal{M}, Z].$$

It follows that $[\mathcal{M}, Z]$ is a σ-algebra.

It is easy to see that $[\mathcal{M}, Z]$ is the smallest σ-algebra containing $\mathcal{M}$ and Z.

The second part of the lemma is an easy consequence of Remark 5 of Lepecki[2].

Theorem 3.1—Let G be complete and $\mu : \mathcal{M} \to G$ be a σ-quasi-measure.

Let $Z \subset X$; for every $\{E_i\} \subset [\mathcal{M}, Z]$, $E_i \downarrow \phi$. and $\{Q_i\} \subset \mathcal{M}$, $Q_i \supset E_i \cap Z^c$, let there be $\{T_i\} \subset \mathcal{M}$, $T_i \downarrow \phi$ such that $Q_i \supset T_i \supset E_i \cap Z^c$.

Then there exists a σ-quasi-measure $\nu : [\mathcal{M}, Z] \to G$ such that $(\mathcal{M}, \mu) \leqslant [\mathcal{M}, Z], \nu)$ and $\mathcal{M}$ is dense in $[\mathcal{M}, Z]$ in the ν-topology.

PROOF : We put $\nu(E) = \mu_*(E \cap Z) + \mu^*(E \cap Z^c)$ for $E \in [\mathcal{M}, Z]$.

Then we shall show that ν has all the required properties.

Let

$$M \in \mathcal{M}.$$

Then

$$\begin{aligned}
\nu(M) &= \mu_*(M \cap Z) + \mu^*(M \cap Z^c) \\
&= \mu_*\{(M \cap Z) \cup (M \cap Z^c)\}, \text{ by Lemma 2.4 (v)} \\
&= \mu_*(M) \\
&= \mu(M), \text{ by Lemma 2.3 (i).}
\end{aligned}$$

Thus ν extends μ.

Let

$$E_i \in [\mathcal{M}, Z],\ i = 1, 2,\ E_1 \cap E_2 = \phi.$$

By Lemma 3.2, $E_i = (M_i \cap Z) \cup (N_i \cap Z^c)$, $i = 1, 2$,

where M_1, M_2 are disjoint and N_1, N_2 are disjoint in $\mathcal{M}$.

Now,

$$\begin{aligned}
\nu(E_1) &= \mu_*(E_1 \cap Z) + \mu^*(E_1 \cap Z^c) \\
&= \mu_*[\{(M_1 \cap Z) \cap Z\} \cup \{(N_1 \cap Z^c) \cap Z\}] \\
&\quad + \mu^*[\{(M_1 \cap Z^c) \cap Z^c\} \cup \{(N_1 \cap Z^c) \cap Z^c\}] \\
&= \mu_*(M_1 \cap Z) + \mu^*(N_1 \cap Z^c).
\end{aligned}$$

Similarly

$$\nu(E_2) = \mu_*(M_2 \cap Z) + \mu^*(N_2 \cap Z^c).$$

Again

$$\begin{aligned}\nu(E_1 \cup E_2) &= \nu[\{(M_1 \cup M_2) \cap Z\} \cup \{(N_1 \cup N_2) \cap Z^c\}] \\ &= \mu_*\{(M_1 \cup M_2) \cap Z\} + \mu^*\{N_1 \cup N_2) \cap Z^c\} \\ &= [\mu_*(M_1 \cap Z) + \mu_*(M_2 \cap Z)] + [\mu^*(N_1 \cap Z^c) \\ &\qquad + \mu^*(N_2 \cap Z^c)], \text{ by Lemma 2.4 (vi), (vii))} \\ &= \nu(E_1) + \nu(E_2).\end{aligned}$$

Thus ν is additive.

Let $\{E_i\} \subset [\mathscr{M}, Z]$ and $E_i \downarrow \phi$. In order to show that ν is σ-additive, it suffices to show that $\nu(E_i) \to 0$.

Let us take W arbitrarily. Then there exists V such that $V + V + V + V \subset W$. Corresponding to $E_i \cap Z$ and V there exists some $P_i \in \mathscr{M}$ such that $P_i \subset E_i \cap Z$ and $\mu_*(E_i \cap Z) \in \mu(P) + V$ for all $P \in \mathscr{M}$ satisfying $P_i \subset P \subset E_i \cap Z$.

Let

$$R_i = \bigcup_{j=i}^{\infty} P_j \subset E_i \cap Z, \text{ since } E_i \cap Z \downarrow .$$

Clearly $\{R_i\}$ is decreasing and $\bigcap_{i=1}^{\infty} R_i = \phi$.

Since $R_i \in \mathscr{M}$ and $P_i \subset R_i \subset E_i \cap Z$, it follows that $\mu_*(E_i \cap Z) \in \mu(R_i) + V$. Again, since μ is σ-additive and exhaustive, $\{R_i\} \subset \mathscr{M}$, $R_i \downarrow \phi$, by Lemma 3 of Lepecki[3], we can find a positive integer i_1 such that $\mu(R_i) \in V$ for $i \geqslant i_1$.

Thus $\mu_*(E_i \cap Z) \in V + V$ for $i \geqslant i_1$.

Again, corresponding to $E_i \cap Z^c$ and V there exists some $Q_i \in \mathscr{M}$ such that $Q_i \supset E_i \cap Z^c$ and $\mu^*(E_i \cap Z^c) \in \mu \subset Q) + V$ for all $Q_i \in \mathscr{M}$ satisfying $Q_i \supset Q \supset E_i \cap Z^c$.

By hypothesis, there exists a decreasing sequence $\{T_i\} \subset \mathscr{M}$ such that $Q_i \supset T_i \subset E_i \cap Z^c$ and $\bigcap_{i=1}^{\infty} T_i = \phi$.

It now follows that $\mu^*(E_i \cap Z^c) \in \mu(T_i) + V$.

But, by Lemma 3 of Lipecki[3] we can find a positive integer i_2 such that $\mu(T_i) \in V$ for $i \geqslant i_2$.

Then $\mu^*(E_i \cap Z^c) \in V + V$ for $i \geqslant i_2$.

Thus $\mu_*(E_i \cap Z) + \mu^*(E_i \cap Z^c) \in V + V + V + V$

$$\subset W \text{ for } i \geqslant i_0,\ i_0 = \max(i_1, i_2).$$

Hence $\nu(E_i) \to 0$. Therefore ν is σ-additive.

The proof of the remaining portion of the theorem is same as that of Lemma 5 of of Lepecki[2].

This completes the proof.

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