ON EXTENSIONS OF $\sigma$-ADDITIVE SET FUNCTIONS WITH VALUES IN A TOPOLOGICAL GROUP

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Two types of extensions of a topological group $(G)$-valued $\sigma$-additive set functions defined on a $\sigma$-algebra $\mathcal{M}$ of subsets of a set $X$ have been discussed. Firstly, extension of a $\sigma$-quasi-measure $\mu : \mathcal{M} \to G$ to a $\sigma$-quasi-measure $\nu : \mathcal{M}^\tau \to G$ has been considered. Next, a $\sigma$-quasi-measure $\mu : \mathcal{M} \to G$ has been extended to a $\sigma$-quasi-measure $\nu : [\mathcal{M}, Z] \to G$, where $Z \subset X$ and $Z \in \mathcal{M}$.

1. INTRODUCTION

This paper discusses the extensions of some results of Lipecki$^3$. He has studied two types of extensions of an additive set function $\mu : \mathcal{M} \to G$, where $\mathcal{M}$ is an algebra of sets and $G$ is an Abelian group. With discrete topology $\delta$ on $G$ he has obtained an extension of $\mu$ to an algebra $\mathcal{M}^\delta$ containing $\mathcal{M}$ from which $\mu$ can not be extended further. In fact, this type of extension has been obtained in case $G$ is a topological semi-group with zero. Also, taking $G$ to be a complete Hausdorff topological group, he has extended a quasi-measure $\mu$ to an algebra $\mathcal{M}^\tau$ containing $\mathcal{M}$, $\tau$ being the topology on $G$ which admits no further extension. Finally, he has proved that a quasi-measure $\mu : \mathcal{M} \to G$, where $G$ is a complete Hausdorff topological group, can be extended to a quasi-measure $\nu$ on the power set $2^X$ so as to ensure denseness of $\mathcal{M}$ in $2^X$ with respect to the topology generated by $\nu$ on $2^X$.

We have proved in the first place that if we take $\mu$ to be $\sigma$-additive and $\mathcal{M}$ to be a $\sigma$-algebra, then $\mu$ admits of an extension to a $\sigma$-additive set function $\nu$ on a $\sigma$-algebra $\mathcal{M}^\delta$ containing $\mathcal{M}$, where $\delta$ is the discrete topology on $G$. This set function $\nu$ admits no further extension. However, this type of extension has been obtained
even when $G$ is a topological semi-group with zero. Further taking $G$ to be a complete Hausdorff topological group, a $\sigma$-quasi-measure $\mu$ has been extended to a $\sigma$-quasi-measure $\nu$ on a $\sigma$-algebra $\mathcal{M}'_\mu$ containing $\mathcal{M}$, where $\tau$ is the topology on $G$, using net limits $\mu^*$ and $\mu_*$. The process of extension of $\mu$ is complete in the sense that no further extension of $\nu$ to a $\sigma$-algebra containing $\mathcal{M}'_\mu$ is possible.

In the second place we have been able to obtain an extension of a $\sigma$-quasi-measure $\mu : \mathcal{M} \rightarrow G$, where $G$ is a complete Hausdorff topological group, to a $\sigma$-quasi-measure $\nu$ on $[\mathcal{M}, Z]$, the $\sigma$-algebra generated by $\mathcal{M}$ and an additional set $Z \subseteq X$ such that $\mathcal{M}$ is dense in $[\mathcal{M}, Z]$ in the $\nu$-topology. The question of its being extended to $2^X$ still remains open.

2. Extension to $\mathcal{M}'_\mu$

Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of a set $X$ and let $(G, +, 0, \tau)$ by an Abelian topological semi-group with zero. Throughout this paper the letters $U, V$ and $W$ sometimes accompanied by indices stand for neighbourhoods of $O$ in the topology $\tau$ in $G$.

Let $\mu : \mathcal{M} \rightarrow G$ be a set function. We denote by $\mathcal{M}'_\mu$ (or briefly by $\mathcal{M}_{\mu}$ when $\tau$ is fixed) the family of all $E \subseteq X$ for which to every $V$ there exist $M, N \in \mathcal{M}$ such that

$$M \subseteq E \subseteq N \text{ and } \mu (S) \in V \text{ is } N \setminus M \supset S \in \mathcal{M}. \quad \ldots (1)$$

**Theorem 2.1** - If $\mu : \mathcal{M} \rightarrow G$ is a $\sigma$-additive set function with $\mu (\phi) = 0$ and $\tau$ is regular, then $\mathcal{M}'_\mu$ is a $\sigma$-algebra of subsets of $X$ containing $\mathcal{M}$.

**Proof:** Suppose $E \in \mathcal{M}'_\mu$. Then to every $V$ there exist $M, N \in \mathcal{M}$ such that $M \subseteq E \subseteq N$ and $\mu (S) \in V$ if $N \setminus M \supset S \in \mathcal{M}$. Clearly $N^c \subseteq E^c \subseteq M^c$ and $N \setminus M = M^c \setminus N^c$. Thus to every $V$ and $E^c$ there exist $M^c, N^c \in \mathcal{M}$ such that $N^c \subseteq E^c \subseteq M^c$ and $\mu (S) \in V$ if $M^c \setminus N^c \supset S \in \mathcal{M}$. Hence $E^c \in \mathcal{M}'_\mu$.

Next, let $E_i \in \mathcal{M}'_\mu$, $i = 1, 2, ...$

Let $U$ be chosen arbitrarily. Then there exists $V$ such that $\overline{V} \subseteq U$. Also, there exist $V_1, V_2, ... , V_i$ such that $V_1 + V_2 + ... + V_i \subseteq V$, $i = 1, 2, ...$. Corresponding to $E_i$ and $V_i$ there exist $M_i, N_i \in \mathcal{M}$ such that $M_i \subseteq E_i \subseteq N_i$ and $\mu (S) \in V_i$ if $N_i \setminus M_i \supset S \in \mathcal{M}$, $i = 1, 2, ...$.
Now,
\[ \bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} N_i. \]

Let
\[ S \subset \bigcup_{i=1}^{\infty} N_i \setminus \bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} (N_i \setminus M_i), \text{ where } S \in \mathcal{H}. \]

We can now write
\[ S = \bigcup_{i=1}^{\infty} S_i' \quad \text{where } S_i' = S \cap (N_i \setminus M_i), \quad S_i' \in \mathcal{H} \]
\[ = \bigcup_{i=1}^{\infty} S_i'' \quad \text{where } \left\{ S_i'' \right\} \text{ is a mutually disjoint sequence} \]

and
\[ S_i'' \subset S_i' \quad , \quad S_i'' \in \mathcal{H}. \]

Then
\[ \mu(S) = \mu \left( \bigcup_{i=1}^{\infty} S_i'' \right) \]
\[ = \sum_{i=1}^{\infty} \mu \left( S_i'' \right) = \text{since } \mu \text{ is } \sigma\text{-additive} \]
\[ = \lim_n \left\{ \sum_{i=1}^{n} \mu \left( S_i'' \right) \right\} \]
\[ = \lim_n \left\{ \mu \left( \bigcup_{i=1}^{n} S_i'' \right) \right\} \]
\[ = \lim_n \left\{ \mu \left( P_n^{*} \right) \right\} \in V \subset U \]

where
\[ P_n^{*} = S_1'' \cup S_2'' \cup \ldots \cup S_n^{*} \]
since
\[ \mu \left( P_n^* \right) \in V_1 + V_2 + \ldots + V_n \subseteq V. \]

It follows that
\[ \bigcup_{i=1}^{\infty} E_i \in M^{\tau}. \]

Also, since \( \mu (\phi) = 0 \), it follows that \( M \subseteq M^{\tau} \).

This completes the proof.

In the following discussion \( \delta \) will denote the discrete topology on \( G \).

It is easy to see that for \( E \in M^{\delta} \), there exist \( M, N \in M \) such that
\[ \mu (S) = 0 \]
whenever \( M \subseteq E \subseteq N \) and \( N \setminus M \supset S \in M \) and that \( M^{\delta} \subseteq M^{\tau} \) for any topology \( \tau \) on \( G \).

**Theorem 2.2**—Let \( \mu : M \rightarrow G \) be \( \sigma \)-additive and \( \mu (\phi) = 0 \). Then the set function \( v : M^{\delta} \rightarrow G \), defined by \( v(E) = \mu (M) \), where \( M, N \in M \) and \( E \) satisfies (1'), is \( \sigma \)-additive.

Moreover,
\[ v | M = \mu \quad \text{and} \quad \left( M^{\delta} \right)^{\delta} = M^{\delta}. \]

**Proof:** As in Theorem 1 of Lipecki\( \delta \) we can show that \( v \) is well-defined on \( M^{\delta} \).

Let \( E_i \in M^{\delta}, i = 1, 2, \ldots, \) and \( E_i \cap E_j = \phi, i \neq j \).

We take
\[ M_i, N_i \in M \] as in (1').

We then have
Let

\[ \bigcup_{i=1}^{\infty} M_i \subset \bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} N_i. \]

Since \( \mu \) is \( \sigma \)-additive, it can be shown that

\[ \mu(S) = \mu\left(S \cap (N_i \setminus M_i)\right) + \mu\left(S \setminus (N_i \setminus M_i)\right) = 0. \]

Also,

\[ \nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i=1}^{\infty} M_i\right) \]

\[ = \sum_{i=1}^{\infty} \mu(M_i) \]

\[ = \sum_{i=1}^{\infty} \nu(E_i). \]

Thus \( \nu \) is \( \sigma \)-additive.

It is easy to see that \( \nu \mid \mathcal{M} = \mu \).

Also, following the arguments of Lipecki (Theorem 1) we can easily show that

\[ \left(\mathcal{M}^x_p\right)^* = \mathcal{M}^x_p. \]

**Theorem 2.3**—Let \( \tau \) be a topology on \( G \) such that there is a countable family \( \{V_i\} \) of neighbourhoods of zero with \( \bigcap_{i=1}^{\infty} 2V_i = \{0\} \). Then for any set function

\[ \mu : \mathcal{M} \rightarrow G, \mathcal{M}^\tau = \mathcal{M}^x_p. \]

Proof is exactly same as that of Proposition 2 of Lipecki.

**Definition 2.1**—A \( \sigma \)-additive set function \( \mu : \mathcal{M} \rightarrow G \) is called a \( \sigma \)-quasi-measure if it is exhaustive. I.e., for any sequence \( \{M_i\} \) of pairwise disjoint sets in \( \mathcal{M} \), \( \mu(M_i) \rightarrow 0 \).
Lemma 2.1—Let $\mu : \mathcal{H} \to G$ be a $\sigma$-quasi-measure. To every $V$ and $E \subset X$, there exist $M, N \in \mathcal{H}$ such that

\begin{align*}
M \subset E \text{ and } \mu (M \setminus M) \in V \text{ if } M \subset M' \subset E, M' \in \mathcal{H}, \\
N \supset E \text{ and } \mu (N \setminus N') \in V \text{ if } E \subset N' \subset N, N' \in \mathcal{H}.
\end{align*}

(2) ...(3)

Proof is exactly same as that of Lemma 1 of Lipecki\(^2\).

For the subsequent discussion we assume that $(G, +, \tau)$ is an Abelian, Hausdorff topological group. Since the topology $\tau$ is fixed, we shall write $\mathcal{H}_\tau$ in place of $\mathcal{H}$.

Lemma 2.2—Let $\mu : \mathcal{H} \to G$ be a $\sigma$-additive set function.

The family $\mathcal{H}_\tau$ consists of all sets $E \subset X$ for which to every $V$ there exist $M, N \in \mathcal{H}$ such that

\begin{align*}
M \subset E \subset N \text{ and } \mu (S') = \mu (S'') \in V
\end{align*}

(1')

if

\begin{align*}
M \subset S', S'' \subset N
\end{align*}

and

\begin{align*}
S', S'' \in \mathcal{H}.
\end{align*}

Proof is exactly same as that of Remark 3 of Lipecki\(^2\).

Let us now assume in addition that $G$ is complete. Let $\mu : \mathcal{H} \to G$ be a $\sigma$-quasi-measure.

For any $E \subset X$ we put

\begin{align*}
\mu_* (E) = \lim \{ \mu (M) : E \supset M \in \mathcal{H} \}
\end{align*}

(4)

and

\begin{align*}
\mu^* (E) = \lim \{ \mu (N) : E \subset N \in \mathcal{H} \},
\end{align*}

(5)

where the sets of indices are directed by inclusion and its converse respectively. Since $G$ is complete and Hausdorff, it follows from Lemma 2.1 that both the limits exist and are unique.

Lemma 2.3—The set functions $\mu_*$ and $\mu^*$ defined by (4) and (5) respectively, have the following properties.
(i) \( \mu(M) = \mu_*(M) = \mu^*(M) \) for \( M \in \mathcal{M} \);

(ii) \( \mu_*(E) = \mu^*(E) \) for \( E \in \mathcal{M}_\mu \);

(iii) \( \mu_*(E_1 \cup E_2) = \mu_*(E_1) + \mu_*(E_2), \ E_1 \cap E_2 = \emptyset, \ E_1, E_2 \in \mathcal{M}_\mu \);

(iv) \( \mu_*(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu_*(E_i) \), where \( \{E_i\} \) is a mutually disjoint sequence of \( \mathcal{M}_\mu \).

**Proof:** (i) It is obvious.

(ii) Let \( E \in \mathcal{M}_\mu \) and \( W \) be chosen arbitrarily. Then there exists \( V \) such that \( V + V - V \subset W \).

Using the definitions of \( \mu_*, \mu^* \) and Lemma 2.2 we can show that

\[ \mu_*(E) - \mu^*(E) \in V + V - V \subset W. \]

Since \( W \) is arbitrary and \( \cap W = 0 \), it follows that \( \mu_*(E) = \mu^*(E) \).

(iii) The proof is exactly same as that of Lemma 2 (iii) of Lepecki.

(iv) Since \( \mu_* \) is additive by (iii), it suffices to show that \( \mu_*(F_i) \to 0 \), where \( \{F_i\} \subset X \) such that \( F_i \downarrow \emptyset \).

Let \( W \) be chosen arbitrarily. We take \( V \) such that \( V + V \subset W \). Corresponding to \( F_i \) and \( V \) there exists \( M_i \in \mathcal{M} \) such that \( M_i \subset F_i \)

and \( \mu_*(F_i) \in \mu(M) + V \) for all \( M \in \mathcal{M} \) satisfying \( M_i \subset M \subset F_i \).

Let

\[ P_i = \bigcup_{i=1}^{\infty} M_i \subset F_i, \ \text{since} \ F_i \downarrow. \]

Clearly \( P_1 \supset P_2 \supset \ldots \) and \( \bigcap_{i=1}^{\infty} P_i = \emptyset \). Thus \( P_i \downarrow \emptyset \).

Since

\[ P_i \in \mathcal{M} \ \text{and} \ M_i \subset P_i \subset F_i, \ \text{it follows that} \ \mu_*(F_i) \in \mu(P_i) + V. \]

Again, since \( \mu \) is \( \sigma \)-additive and exhaustive, and \( \{P_i\} \subset \mathcal{M}, \ P_i \downarrow \emptyset \), by Lemma 3, of Lepecki we can find a positive integer \( i_0 \) such that \( \mu(P_i) \in V \) for \( i \geq i_0 \).

Consequently, \( \mu_*(F_i) \in V + V \subset W \) for \( i \geq i_0 \).
Thus $\mu^*_\sigma(F_1) \to 0$. Hence $\mu^*$ is $\sigma$-additive on $\mathcal{M}_\mu$.

**Definition 2.2**—A $\sigma$-additive set function $\mu : \mathcal{M} \to G$ is called $\mathcal{K}$-tight ($\mathcal{K}$ being a sub-family of $\mathcal{M}$) if for every $M \in \mathcal{M}$ and $V$, there exists a $K \in \mathcal{K}$ such that $K \subset M$ and $\mu(K) - \mu(M) \in V$ whenever $K \subset M \subset M'$, $M' \in \mathcal{M}$.

**Theorem 2.4**—Let $G$ be an Abelian, complete Hausdorff topological group. Every $\sigma$-quasi-measure $\mu : \mathcal{M} \to G$ has the unique extension to an $\mathcal{M}$-tight $\sigma$-quasi-measure $\nu : \mathcal{M}_\mu \to G$.

Moreover, $(\mathcal{M}_\mu)_\nu = \mathcal{M}_\mu$, $\nu^* = \mu^*$ and $\nu^* = \mu^*$.

**Proof:** We put $\nu(E) = \mu^*(E)$ for $E \in \mathcal{M}_\mu$.

Let $M \in \mathcal{M}$. Then $\nu(M) = \mu^*(M) = \mu(M)$, by Lemma 2.3 (i).

Thus $\nu$ is an extension of $\mu$.

Let

\[ \{E_i\} \subset \mathcal{M}_\mu \text{ and } E_i \cap E_j = \emptyset, i \neq j, \ i, j = 1, 2, \ldots \]

Suppose

\[ E = \bigcup_{i=1}^{\infty} F_i. \text{ Then } E \in \mathcal{M}_\mu. \]

Now,

\[ \nu(E) = \mu^* \left( \bigcup_{i=1}^{\infty} E_i \right) \]

\[ = \sum_{i=1}^{\infty} \mu^*(E_i), \text{ by Lemma 2.3 (iv)} \]

\[ = \sum_{i=1}^{\infty} \nu(E_i). \]

Thus $\nu$ is $\sigma$-additive.

Using the definitions of $\mu^*$ and $\nu$ we can show that $\nu$ is $\mathcal{M}$-tight. The exhaustivity of $\nu$ follows easily from $\mathcal{M}$-tightness of $\nu$ and exhaustivity of $\mu$. The uniqueness of $\nu$ is also an easy consequence of its $\mathcal{M}$-tightness.

We shall now prove that $(\mathcal{M}_\mu)_\nu = \mathcal{M}_\mu$. 

Let $W$ be chosen arbitrarily. We take $V$ such that $V + V + V \subset W$. Let $F \in (\mathcal{M}_\mu)_\nu$. Then by definition, there exist $E_1, E_2 \in \mathcal{M}_\mu$ such that $E_1 \subset F \subset E_2$ and $\nu(S) \in V$ if $E_2 \setminus E_1 \supset S \neq \mathcal{M}_\gamma$. In particular, $(R) \in V$ for $E_1 \setminus E_1 \supset R \in \mathcal{M}$. We take $M_i, N_i \in \mathcal{M}$ such that $M_i \subset E_i \subset N_i$, $i = 1, 2$ and $\mu(S) \in V$ if $N_i \setminus M_i \supset S \in \mathcal{M}$, $i = 1, 2$. We have $M_i \subset F \subset N_i$ and $N_i \setminus M_i \subset (N_i \setminus M_i) \cup (E_i \setminus E_1) \cup (N_i \setminus M_i)$. Let $S \in \mathcal{M}$ and $S \subset N_i \setminus M_i$. It now follows by the additivity of $\mu$ that $\mu(S) \in W$. Thus $F \in \mathcal{M}_\mu$.

Consequently, $(\mathcal{M}_\mu)_\nu \subset \mathcal{M}_\mu$.

Also, it is obvious that $\mathcal{M}_\mu \subset (\mathcal{M}_\mu)_\nu$.

Thus $(\mathcal{M}_\mu)_\nu = \mathcal{M}_\mu$.

Next, we shall show that $\nu^* = \mu^*$.

Let $E \subset X$. We have $\nu^*(E) = \lim \nu(M) : E \supset M \in \mathcal{M}_\mu$

$= \lim \mu^*(M) : E \supset M \in \mathcal{M}_\mu$.

Then there exists $M_0 \in \mathcal{M}_\mu$ such that $\nu^*(E) = \mu^*(M) \in V$ for all $M \in \mathcal{M}_\mu$ satisfying $M_0 \subset M \subset E$.

In particular, $\nu^*(E) = \mu(M) \in V$ for all $M \in \mathcal{M}$ satisfying $M_0 \subset M \subset E$.

Let $M_0 \subset N_0 \subset E$, where $N_0 \in \mathcal{M}$.

Then $\nu^*(E) = \mu(M) \in V$ for all $M \in \mathcal{M}$ satisfying $M_0 \subset M \subset E$.

Thus $\nu^*(E) = \lim \mu(M) : E \supset M \in \mathcal{M}$,

$= \mu^*(E)$.

Consequently, $\nu^* = \mu^*$.

Analogously we can prove that $\nu^* = \mu^*$.

This completes the proof.

**Lemma 2.4**—Let $\mu_\mu$ and $\mu^*$ be the same as in Lemma 2.3. Then

(v) $\mu_\mu(A) + \mu^*(B) = \mu_\mu(A \cup B)$ for $A, B \subset X$ such that $A \cup B \in \mathcal{M}_\mu$ and $A \cap B = \emptyset$;

(vi) $\mu_\mu(A) + \mu_\mu(B) = \mu_\mu(A \cup B)$ for $A, B \subset X$ such that there exists an $E \in \mathcal{M}_\mu$ with $A \subset E$ and $B \cap E = \emptyset$;

(vii) $\mu^*(A) + \mu^*(B) = \mu^*(A \cup B)$ for $A, B \subset X$ such that there exists an $E \in \mathcal{M}_\mu$ with $A \subset E$ and $B \cap E = \emptyset$. **
PROOF: In view of Theorem 2.4, it is enough to prove all the above statements with \( \mathcal{M}_\ast \) replaced by \( \mathcal{M} \). Proofs of (v) and (vi) are exactly same as those of Lemma 3 (iv), (v) of Lipecki. For the proof of (vii) we choose arbitrarily and take \( V \) such that \( V - V - V \subseteq W \).

Using the definition of \( \mu^* \) and the given condition we can show that

\[
\mu^* (A \cup B) = \mu^* (A) + \mu^* (B) - \mu^* (A \cap B) \subseteq V - V - V \subseteq W.
\]

It follows that \( \mu^* (A \cup B) = \mu^* (A) + \mu^* (B) \).

Following the arguments of Lipecki, we can establish the following corollary:

**Corollary**—Let \( E \subseteq X \) be an arbitrary set. The following conditions are equivalent:

(a) \( E \in \mathcal{M}_\ast \);
(b) \( \mu^* (Z) = \mu^* (Z \cap E) + \mu^* (Z \setminus E) \) for all \( Z \subseteq X \);
(c) \( \mu^* (Z) = \mu^* (Z \cap E) + \mu^* (Z \setminus E) \) for all \( Z \subseteq X \);
(d) \( \mu (S) = \mu^* (S \cap E) + \mu^* (S \setminus E) \) for all \( S \in \mathcal{M} \);
(e) \( \mu (S) = \mu^* (S \cap E) + \mu^* (S \setminus E) \) for all \( S \in \mathcal{M} \).

3. **Extension to \([\mathcal{M}, E]\)**

Let \( \mathcal{M} \) be an algebra of subsets of \( X \).

Then it can be considered as a group with the symmetric difference of sets \( \Delta \) as group operation.

Let \( \mu : \mathcal{M} \to G \) be an additive set function.

Then \( \mu \) gives rise to a topology on \( \mathcal{M} \) called the \( \mu \)-topology whose neighbourhood base at \( \phi \) is given by \( \{ M \in \mathcal{M} : \mu (S) \subseteq V \text{ for all } M \supseteq S \in \mathcal{M}, \} \), where \( V \) runs through some neighbourhood base at zero in \( G \). (Proposition 1.9 of Drewnowski.)

**Lemma 3.1**—Let \( \mathcal{M} \) and \( \mathcal{N} \) be two \( \sigma \)-algebras of subsets of \( X \) and \( \mathcal{M} \subseteq \mathcal{N} \). Let \( \nu : \mathcal{N} \to G \) be \( \sigma \)-additive and \( \mathcal{M} \) be a dense sub-\( \sigma \)-algebra in the \( \nu \)-topology. If \( \nu \) restricted to \( \mathcal{M} \) is exhaustive, then it is also exhaustive on \( \mathcal{N} \).

Proof is similar to that of Lemma 4 of Lepecki.

Let \( \mathcal{M} \) and \( \mathcal{N} \) be \( \sigma \)-algebras of subsets of \( X \) and let \( \mu : \mathcal{M} \to G \) and \( \nu : \mathcal{N} \to G \) be \( \sigma \)-additive. We write \((\mathcal{M}, \mu) \leq (\mathcal{N}, \nu)\) if and only if \( \mathcal{M} \subseteq \mathcal{N} \), \( \nu | \mathcal{M} = \mu \).
and for any closed neighbourhood $V$ of zero and $M \in \mathcal{M}$ such that $\mu(S) \in V$ whenever $M \supset S \in \mathcal{M}$, we have $v(N) \in V$ if $M \supset N \in \mathcal{M}$.

It is clear that the relation $\ll$ is a partial order and that if $(\mathcal{M}, \mu) \ll (\mathcal{M}, v)$, then the $\mu$-topology coincides with the topology induced on $\mathcal{M}$ by the $v$-topology.

If $\mathcal{M}$ be a family of subsets of $X$, $Z \subset X$, we denote by $[\mathcal{M}, Z]$ the family of all sets of the form $(M \cap Z) \cup (N \cap Z^c)$, where $M, N \in \mathcal{M}$.

**Lemma 3.2**—If $\mathcal{M}$ is a $\sigma$-algebra, then $[\mathcal{M}, Z]$ is the smallest $\sigma$-algebra containing $\mathcal{M}$ and $Z$.

Moreover, for any disjoint $E_1, E_2 \in [\mathcal{M}, Z]$, there exist disjoint $M_1, M_2$ and disjoint $N_1, N_2$ in $\mathcal{M}$ such that $E_i = (M_i \cap Z) \cup (N_i \cap Z^c)$, $i = 1, 2$.

**Proof:** For the proof of the first part of the lemma we proceed as follows:

$[\mathcal{M}, Z]$ is an algebra (p. 268).

Let

$$\{E_n\} \subset [\mathcal{M}, Z].$$

Now,

$$E_n = (M_n \cap Z) \cup (N_n \cap Z^c), \text{ where } M_n, N_n \in \mathcal{M}.$$ 

$$\bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} [(M_n \cap Z) \cup (N_n \cap Z^c)]$$

$$= [\bigcup_{n=1}^{\infty} (M_n \cap Z)] \cup [\bigcup_{n=1}^{\infty} (N_n \cap Z^c)]$$

$$= [(\bigcup_{n=1}^{\infty} M_n) \cap Z] \cup [(\bigcup_{n=1}^{\infty} N_n) \cap Z^c].$$

Thus

$$\bigcup_{n=1}^{\infty} E_n \in [\mathcal{M}, Z].$$

It follows that $[\mathcal{M}, Z]$ is a $\sigma$-algebra.

It is easy to see that $[\mathcal{M}, Z]$ is the smallest $\sigma$-algebra containing $\mathcal{M}$ and $Z$. 
The second part of the lemma is an easy consequence of Remark 5 of Lepecki\textsuperscript{2}.

**Theorem 3.1**—Let $G$ be complete and $\mu : \mathcal{M} \rightarrow G$ be a $\sigma$-quasi-measure.

Let $Z \subset X$; for every $\{E_i\} \subset [\mathcal{M}, Z]$, $E_i \downarrow \phi$. and $\{Q_i\} \subset \mathcal{M}$, $Q_i \supset E_i \cap Z^c$, let there be $\{T_i\} \subset \mathcal{M}$, $T_i \downarrow \phi$ such that $Q_i \supset T_i \supset E_i \cap Z^c$.

Then there exists a $\sigma$-quasi-measure $\nu : [\mathcal{M}, Z] \rightarrow G$ such that $(\mathcal{M}, \mu) \leq [\mathcal{M}, Z], \nu$ and $\mathcal{M}$ is dense in $[\mathcal{M}, Z]$ in the $\nu$-topology.

**Proof:** We put $\nu(E) = \mu_\# (E \cap Z) + \mu^* (E \cap Z^c)$ for $E \in [\mathcal{M}, Z]$.

Then we shall show that $\nu$ has all the required properties.

Let

$$M \in \mathcal{M}.$$  

Then

$$\nu(M) = \mu_\# (M \cap Z) + \mu^* (M \cap Z^c)$$

$$= \mu_\# \{(M \cap Z) \cup (M \cap Z^c)\}, \text{ by Lemma 2.4 (v)}$$

$$= \mu_\# (M)$$

$$= \mu(M), \text{ by Lemma 2.3 (i)}.$$  

Thus $\nu$ extends $\mu$.

Let

$$E_i \in [\mathcal{M}, Z], \ i = 1, 2, \ E_1 \cap E_2 = \phi.$$  

By Lemma 3.2, $E_i = (M_i \cap Z) \cup (N_i \cap Z^c), \ i = 1, 2$, where $M_i, M_2$ are disjoint and $N_1, N_2$ are disjoint in $\mathcal{M}$.

Now,

$$\nu(E_i) = \mu_\# (E_i \cap Z) + \mu^* (E_i \cap Z^c)$$

$$= \mu_\# \left[\left\{(M_1 \cap Z) \cap Z \cup \{(N_1 \cap Z^c) \cap Z\}\right\}\right]$$

$$+ \mu^* \left[\left\{(M_1 \cap Z^c) \cap Z^c \cup \{(N_1 \cap Z^c) \cap Z^c\}\right\}\right]$$

$$= \mu_\# (M_1 \cap Z) + \mu^* (N_1 \cap Z^c).$$
Similarly

\[ \nu(E_2) = \mu^\ast(M_2 \cap Z) + \mu^\ast(N_2 \cap Z^c). \]

Again

\[ \nu(E_1 \cup E_2) = \nu[(M_1 \cup M_2) \cap Z] \cup [(N_1 \cup N_2) \cap Z^c] \]
\[ = \mu^\ast((M_1 \cup M_2) \cap Z) + \mu^\ast(N_1 \cup N_2) \cap Z^c \]
\[ = [\mu^\ast(M_1 \cap Z) + \mu^\ast(M_2 \cap Z)] + [\mu^\ast(N_1 \cap Z^c) \]
\[ + \mu^\ast(N_2 \cap Z^c)], \text{ by Lemma 2.4 (vi), (vii)} \]
\[ = \nu(E_1) + \nu(E_2). \]

Thus \( \nu \) is additive.

Let \( \{E_i\} \subset [\mathcal{H}, Z] \) and \( E_i \downarrow \phi \). In order to show that \( \nu \) is \( \sigma \)-additive, it suffices to show that \( \nu(E_i) \to 0 \).

Let us take \( W \) arbitrarily. Then there exists \( V \) such that \( V + V + V + V \subset W \). Corresponding to \( E_i \cap Z \) and \( V \) there exists some \( P_i \in \mathcal{H} \) such that \( P_i \subset E_i \cap Z \) and \( \mu^\ast(E_i \cap Z) \in \mu(P) + V \) for all \( P \in \mathcal{H} \) satisfying \( P_i \subset P \subset E_i \cap Z \).

Let

\[ R_i = \bigcup_{j=i}^{\infty} P_j \subset E_i \cap Z, \text{ since } E_i \cap Z \downarrow. \]

Clearly \( \{R_i\} \) is decreasing and \( \bigcap_{i=1}^{\infty} R_i = \phi \).

Since \( R_i \in \mathcal{H} \) and \( P_i \subset R_i \subset E_i \cap Z \), it follows that \( \mu^\ast(E_i \cap Z) \in \mu(R_i) + V \).

Again, since \( \mu \) is \( \sigma \)-additive and exhaustive, \( \{R_i\} \subset \mathcal{H}, R_i \downarrow \phi \), by Lemma 3 of Lepecki\(^3\), we can find a positive integer \( i_1 \) such that \( \mu(R_i) \in V \) for \( i \geq i_1 \).

Thus \( \mu^\ast(E_i \cap Z) \in V + V \) for \( i \geq i_1 \).

Again, corresponding to \( E_i \cap Z^c \) and \( V \) there exists some \( Q_i \in \mathcal{H} \) such that \( Q_i \supset E_i \cap Z^c \) and \( \mu^\ast(E_i \cap Z^c) \in \mu \subset Q) + V \) for all \( Q_i \in \mathcal{H} \) satisfying \( Q_i \supset Q \supset E_i \cap Z^c \).
By hypothesis, there exists a decreasing sequence $\{T_i\} \subseteq \mathcal{T}$ such that $Q \supseteq T_i \subseteq E_i \cap Z^c$ and $\bigcap_{i=1}^{\infty} T_i = \emptyset$.

It now follows that $\mu^*(E_i \cap Z^c) \subseteq \mu(T_i) + V$.

But, by Lemma 3 of Lepecki$^3$ we can find a positive integer $i_2$ such that $\mu(T_i) \subseteq V$ for $i \geq i_2$.

Then $\mu^*(E_i \cap Z^c) \subseteq V + V$ for $i \geq i_2$.

Thus $\mu^*(E_i \cap Z) + \mu^*(E_i \cap Z^c) \subseteq V + V + V + V$.

Hence $\nu(E_i) \to 0$. Therefore $\nu$ is $\sigma$-additive.

The proof of the remaining portion of the theorem is same as that of Lemma 5 of Lepecki$^3$.

This completes the proof.

References