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# APPROXIMATE SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS

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In this paper sufficient conditions have been obtained for the existence of a best approximate solution of degree $k$ of delay differential equations

$$(*) \quad \frac{dx}{dt} = F(t, x(t), x(t-\sigma)), \; t \in [0, a]$$

$$x(t) = \varphi(t), \; t \in [-\sigma, 0].$$

If $x(t)$ is the unique solution of (*) on $[0, a]$ and $f_k^0(t)$ is a best approximate solution of degree $k$ of (*) then it has been proved that $\lim_{k \to \infty} \| f_k^0 - x \| = 0$. The bounds of the error have been obtained and an example has been given to illustrate the theory.

§ 1. A good deal of work has been done on the question of finding approximate solutions of initial value problem of ordinary differential equations. For this, the reader is referred to Pacapoulous and Kartsatos[2], Henry[5] and Huffstutler and Stein[7] and the references therein. The problem of finding approximate solutions of initial value problem of differential equations with deviating arguments was considered by Allinger and Henry[1] and of functional differential equations was considered by Henry[6] and Lim[10]. In their works the initial set $E_0 = \{0\}$ and the deviation is of the type $t/2$.

In this paper we find approximate solutions of initial value problem of delay-differential equations of the type

$$\frac{dx}{dt} = F(t, x(t), x(t-\sigma)), \quad t \in [0, a] \qquad \ldots(1)$$

$$x(t) = \varphi(t), \; t \in [-\sigma, 0] \qquad \ldots(2)$$

where $\sigma > 0$, $a > 0$, $\varphi : [-\sigma, 0] \to R^n$ is continuous, $F : [0, a] \times R^n \times R^n \to R^n$ and $R^n$ is the $n$-dimensional Euclidean space. Here the initial set $E_0 = [-\sigma\ 0]$.

It is clear that $x(t)$ is a solution of (1) on $[0, a]$ with initial function $\varphi(t)$ on $[-\sigma, 0]$ if and only if

$$x(t) = \varphi(t), \quad t \in [-\sigma, 0]$$

$$x(t) = \varphi(0) + \int_0^t F(s, x(s), x(s-\sigma))\, ds, \quad t \in [0, a].$$

For a continuous function $F$, we define

$T : C[-\sigma, a] \to C[-\sigma, a]$ and $S : C[-\sigma, a] \to C[-\sigma, a]$ as follows :

$$Tx(t) = \begin{cases} \varphi(t), & t \in [-\sigma, 0] \\ \varphi(0) + \int_0^t F(s, x(s), x(s), x(s-\sigma)), & t \in [0, a] \end{cases}$$

and

$$S\,x(t) = x(t) - Tx(t), \quad t \in [-\sigma, a]$$

for

$$x \in C[-\sigma, a], \text{ where } C[-\sigma, a] \equiv C([-\sigma, a], R^n)$$

is the space of continuous functions with domain $[-\sigma, a]$ and range in $R^n$. So $x(t)$ is a solution of (1) and (2) on $[0, a]$ if and only if $Tx(t) = x(t)$, $t \in [-\sigma, a]$. For $x \in C[-\sigma, a]$, we define

$$\|x\| = \sup_{t \in I} |x(t)|,$$

where $I = [-\sigma, a]$.

Let $P_k$ be the set of all functions $f_k : [-\sigma, a] \to R^n$ such that

$$f_k(t) = \begin{cases} \varphi(t), & t \in [-\sigma, 0] \\ P_k(t), & t \in [0, a], \end{cases}$$

where $p_k(t)$ is a polynomial in $t \in [0, a]$ with coefficients in $R^n$ and of degree less than or equal to $k$ such that $p_k(0) = \varphi(0)$. So $P_k \subset C[-\sigma, a]$. If there exists a function $\bar{f}_k \in p_k$ such that

$$\|S\bar{f}_k\| = \inf_{f_k \in p_k} \|S f_k\|,$$

then $\bar{f}_k$ is said to be a best approximate solution of (1) and (2) of degree $k$.

In this paper we find conditions on $F$ under which (1) and (2) admit a best approximate solution $f_k^0$ of degree $k$. Next we show that $\|f_k^0 - x\| \to 0$ as $k \to \infty$, where $x(t)$ is the unique solution of (1) and (2). Also the bounds of the error have been found. Finally an example has been given to illustrate the theory. The problem of local existence of the unique solution of (1) and (2) has been considered in Driver[3] and El'sgol'c and Norkin[4]. In general, in theorems concerning local existence of the unique solution of the initial value problem (1) and (2), the condition is imposed on the length of the interval of existence. In particular, this is so when we apply the contraction mapping principle for the existence problem. This condition enables one to show that the operator $T$ is a contraction. In this paper we accomplish this without imposing any restriction on the length of the interval of existence. This becomes possible because of the introduction of a new type of norm on $C[-\sigma, a]$.

Suppose that $F$ satisfies following conditions :

$(H_1)$ $F$ is continuous

and

$(H_2)$ there exists a constant $L > 0$ such that

$$| F(t, u_1, v_1) - F(t, u_2, v_2) |$$

$$\leqslant L [ | u_1 - u_2 | + | v_1 - v_2 | ]$$

for $(t, u_1, v_1)$ and $(t, u_2, v_2) \in [0, a] \times R^n \times R^n$.

For $\lambda = 3L$ and $g \in C[-\sigma, a]$, we define

$$\|g\|^\lambda = \sup_{t \in I} \{ | g(t) | e^{-\lambda t} \}.$$

So

$$e^{-\lambda a} \|g\| \leqslant \|g\|^\lambda \leqslant e^{\lambda \sigma} \|g\|. \qquad \ldots (3)$$

From (3) it is clear that $C[-\sigma, a]$ is a Banach space with respect to the norm $\|.\|^\lambda$.

§ 2. In this section we prove a theorem on the existence of the unique solution of (1) and (2) on $[0, a]$. No restriction has been imposed on the length of the interval of existence $[0, a]$.

*Theorem* 1—Suppose that $F$ satisfies hypotheses $(H_1)$ and $(H_2)$. Then there exists a unique solution of (1) and (2) on $[0, a]$.

PROOF : Define $\bar{\varphi} : [-\sigma, a] \to R^n$ as follows :

$$\bar{\varphi}(t) = \begin{cases} \varphi(t), t \in [-\sigma, 0] \\ \varphi(0), t \in [0, a]. \end{cases}$$

Let $\Omega = \{y \in C[-\sigma, a] : y(t) = \varphi(t), t \in [-\sigma, 0]\}$. As $\bar{\varphi} \in \Omega$, $\Omega \neq \phi$. Clearly, $\Omega$ is a closed subset of $C[-\sigma, a]$ with respect to the norm $\|.\|^{\lambda}$ and hence is complete. From the definition of $T$ it follows that $T\Omega \subset \Omega$, that is, $T : \Omega \to \Omega$.

Next we show that $T$ is a contraction. For $x_1$ and $x_2 \in \Omega$,

$$\begin{aligned} &| Tx_1(t) - Tx_2(t) | \\ &\leqslant \int_0^t | F(s, x_1(s), x_1(s-\sigma)) - F(s, x_2(s), x_2(s-\sigma)) | \, ds \\ &\leqslant L \left[\int_0^t | x_1(s) - x_2(s) | \, ds + \int_0^t | x_1(s-\sigma) - x_2(s-\sigma) | \, ds\right] \\ &\leqslant 2L \int_{-\sigma}^t | x_1(s) - x_2(s) | \, ds \\ &\leqslant 2L \int_{-\sigma}^t | x_1(s) - x_2(s) | \, e^{-\lambda s} e^{\lambda s} \, ds \\ &\leqslant 2L \, \|x_1 - x_2\|^{\lambda} \int_{-\sigma}^t e^{\lambda s} \, ds \\ &\leqslant \frac{2L}{\lambda} e^{\lambda t} \, \|x_1 - x_2\|^{\lambda}. \end{aligned}$$

So

$$\|Tx_1 - Tx_2\|^{\lambda} \leqslant \frac{2L}{\lambda} \|x_1 - x_2\|^{\lambda}. \qquad \ldots(4)$$

Hence $T$ is a contraction. Consequently, $T$ has a unique fixed point $x \in \Omega$. This $x$ is the required unique solution of (1) and (2) on $[0, a]$.

Hence the theorem.

*Remark* : We may note that (4) holds for $x_1$ and $x_2 \in C[-\sigma, a]$.

§ 3. This section is concerned with the existence of a best approximate solution $f_k^0$ of degree $k$ of (1) and (2). Also we show that $\|f_k^0 - x\| \to 0$ as $k \to \infty$, where $x(t)$ is the unique solution of (1) and (2) on $[0, a]$.

*Theroem* 2—If $F$ satisfies $(H_1)$ and $(H_2)$, then for every integer $k > 0$ there exists a best approximate solution of (1) and (2) of degree $k$.

PROOF : Define

$$g_0(t) = \begin{cases} \varphi(t), & t \in [-\sigma, 0] \\ \varphi(0), & t \in [0, a]. \end{cases}$$

So $g_0 \in p_k$. Set $r = 2\|Sg_0\|$ and $w = \inf_{f_k \in p_k} \|Sf_k\|$. Hence $r > 0$. Setting

$$A = \{f_k \in p_k : \|Sf_k\| \leqslant r\}$$

we see that $A \neq \phi$ because $g_0 \in A$. Clearly, $w = \inf_{f_k \in A} \|Sf_k\|$. So there exists a sequence $\langle f_k^n \rangle \subset A$ such that

$$w = \lim_{n \to \infty} \|Sf_k^n\|. \qquad \ldots(5)$$

Let $x(t)$ be the unique solution of (1) and (2) on $[0, a]$. For $f_k \in A$, we have, by using (4),

$$\|f_k - x\|^\lambda \leqslant \|f_k - Tf_k\|^\lambda + \|Tf_k - Tx\|^\lambda$$

$$\leqslant \|Sf_k\|^\lambda + \frac{2L}{\lambda} \|f_k - x\|^\lambda$$

$$\leqslant \|Sf_k\|^\lambda + \frac{2}{3} \|f_k - x\|^\lambda.$$

So

$$\|f_k - x\|^\lambda \leqslant 3\|Sf_k\|^\lambda. \qquad \ldots(6)$$

From (3) and (6) we have

$$\|f_k\| \leqslant e^{\lambda a} \|f_k\|^\lambda$$

$$\leqslant e^{\lambda a} [\|f_k - x\|^\lambda + \|x\|^\lambda]$$

$$\leqslant e^{\lambda a} [3\|Sf_k\|^\lambda + \|x\|^\lambda]$$

$$\leqslant e^{\lambda(a+\sigma)} [3\|Sf_k\| + \|x\|]$$

$$\leqslant e^{\lambda(a+\sigma)} [3r + \|x\|].$$

So the set $A$ is bounded. Define

$$B = \{g_k : g_k = f_k \mid [0, a] \text{ and } f_k \in A\}.$$

Hence

$$B = \{p_k : \|S f_k\| \leqslant r\}.$$

Since

$$\|g_k\|_{[0,a]} = \|f_k\|_{[0,a]} \leqslant \|f_k\|.$$

then $B$ is bounded, where $\|g_k\|_{[0,a]} = \sup_{t\in[0,a]} |g_k(t)|$. As the set of all polynomials in $t \in [0, a]$ with coefficients in $R^n$ and of degree $\leqslant k$ along with the zero polynomial forms $a(k+1)$-dimensional normed linear space with respect to the sup norm and since $B$ is a bounded subset of this finite dimensional normed linear space, then $B$ is compact[9]. Let $g_k^n = f_k^n \mid [0, a]$. So $<g_k^n> \subset B$. $B$ compact implies that there exists a subsequence $<g_k^{n_i}>$ of $<g_k^n>$ such that $\|g_k^{n_i} - g_k^0\|_{[0,a]} \to 0$ as $n_i \to \infty$, where $g_k^0 \in B$. So there exists $f_k^0 \in A$ such that $g_k^0 = f_k^0 \mid [0, a]$. Corresponding to the subsequence $<g_k^{n_i}>$, consider the subsequence $<f_k^{n_i}>$ of $<f_k^n>$. Since

$$\|f_k^{n_i} - f_k^0\| = \|f_k^{n_i} - f_k^0\|_{[0,a]} = \|g_k^{n_i} - g_k^0\|_{[0,a]}$$

then

$$\lim_{n_i\to\infty} \|f_k^{n_i} - f_k^0\| = 0.$$

From the continuity of $S$ and (5), we obtain

$$\|Sf_k^0\| = \lim_{n_i\to\infty} \|Sf_k^{n_i}\| = w = \inf_{f_k\in p_k} \|Sf_k\|.$$

So $f_k^0$ is a best approximate solution of (1) and (2) of degree $k$. This completes the proof of the theorem.

*Theorem* 3—Let $F$ satisfy the hypotheses $(H_1)$ and $(H_2)$. If $x(t)$ is the unique solution of (1) and (2) on $[0, a]$ and if $f_k^0$ is a best approximate solution of (1) and (2) of degree $k$, then $\lim_{k\to\infty} \|f_k^0 - x\| = 0$.

PROOF : Set $y = x \mid [0, a]$. So $y : [0, a] \to R^n$ is continuous. Let $\epsilon > 0$. Choose a positive integer $n$ such that

$$3(1 + 2aL)\, e^{\lambda(\sigma+a)} \leqslant n\epsilon.$$

It is possible to construct a polynomial $q^*(t)$ in $t \in [0, a]$ of degree $N$ such that $q^*(0) = y(0)$ and

$$\|y - q^*\|_{[0,a]} \leqslant \frac{1}{n}$$

(see Isaacson and Keller[8], pp. 184 and 185). Define

$$q(t) = \begin{cases} \varphi(t), & t \in [-\sigma, 0] \\ q^*(t), & t \in [0, a]. \end{cases}$$

So $q \in P_N \subset C[-\sigma, a]$,

For $t \in [0, a]$,

$$\begin{aligned} &| Tq(t) - Tx(t) | \\ &\leqslant \int_0^t | F(s, q(s), q(s-\sigma)) - F(s, x(s), x(s-\sigma)) |\, ds \\ &\leqslant L \int_0^t [\, | q(s) - x(s) | + | q(s-\sigma) - x(s-\sigma) |\,]\, ds \\ &\leqslant 2aL \|q - x\| \\ &\leqslant 2aL \|q^* - y\|_{[0,a]} \\ &\leqslant \frac{2aL}{n}. \end{aligned}$$

Hence

$$\| Tq - Tx \| \leqslant \frac{2aL}{n}.$$

From Theorem 2 it follows that there exists $f_N^0 \in p_N$ such that

$$\| Sf_N^0 \| = \inf_{f_N \in p_N} \| Sf_N \|$$

and for positive integer $k \geq N$ there exists $f_k^0 \in p_k$ such that

$$\| Sf_k^0 \| = \inf_{f_k \in p_k} \| Sf_k \|.$$

Proceeding as in Theorem 2 we obtain

$$\| f_k^0 - x \|^\lambda \leq 3 \| Sf_k^0 \|^\lambda$$

$$\leq 3 e^{\lambda\sigma} \| Sf_k^* \|$$

$$\leq 3 e^{\lambda\sigma} \| Sf_N^0 \|$$

$$< 3 e^{\lambda\sigma} \| Sq \|$$

because $q \in P_N \subset P_k$. Further,

$$\| Sq \| = \| q - Tq \|$$

$$< \| q - x \| + \| Tx - Tq \|$$

$$\leq \| q^* - y \|_{[0,a]} + \| Tx - Tq \|$$

$$< \frac{1}{n} + \frac{2aL}{n}.$$

Hence

$$\| f_k^0 x \| < e^{\lambda a} \| f_k^0 - x \|^\lambda$$

$$\leq 3 e^{\lambda(a+\sigma)} \frac{(2aL + 1)}{n}$$

$$\leq \epsilon.$$

This proves that $\lim_{k \to \infty} \| f_k^* - x \| = 0$ and hence the theorem.

§ 4. In this section we find the bounds of the error and give an example to illustrate the theory.

Suppose that $F$ satisfies the hypotheses $(H_1)$ and $(H_2)$. Let $f_k^0$ be a best approximate solution of (1) and (2) of degree $k$ and let $x(t)$ be the unique solution of (1) and (2) on $[0, a]$. Now

$$\| x - f_k^0 \| \leqslant e^{\lambda a} \| x - f_k^0 \|^{\lambda}$$

$$\leqslant 3\, e^{\lambda a} \| S f_k^0 \|^{\lambda}$$

$$\leqslant 3\, e^{\lambda(a+\sigma)} \| S f_k^0 \|.$$

Further, from (4) it follows that

$$\| x - f_k^0 \|^{\lambda} = \| Tx - Tf_k^0 + Tf_k^0 - f_k^0 \|^{\lambda}$$

$$\geqslant \| Tf_k^0 - f_k^0 \|^{\lambda} - \| Tx - Tf_k^0 \|^{\lambda}$$

$$\geqslant \| S f_k^0 \|^{\lambda} - \frac{2}{3} \| x - f_k^0 \|^{\lambda}.$$

So

$$\| x - f_k^0 \|^{\lambda} \geqslant \frac{3}{5} \| S f_k^0 \|^{\lambda}$$

$$\geqslant \frac{3}{5} e^{-\lambda a} \| S f_k^0 \|.$$

Hence

$$\| x - f_k^0 \| \geqslant e^{-\lambda a} \| x - f_k^0 \|^{\lambda}$$

$$\geqslant \frac{3}{5} e^{\lambda(a+\sigma)} \| S f_k^0 \|.$$

Consequently,

$$\frac{3}{5} e^{-\lambda(a+\sigma)} \| S f_k^0 \| \leqslant \| x - f_k^0 \| \leqslant 3\, e^{\lambda(a+\sigma} \| S f_k^0 \|.$$

*Example*—Consider

$$\frac{dx}{dt} = -x \left( t - \frac{\pi}{2} \right), \quad t \in \left[ 0, \frac{3\pi}{2} \right] \qquad \ldots(7)$$

$$x(t) = \sin t,\ t \in \left[-\frac{\pi}{2}, 0\right]. \qquad \ldots(5)$$

Here $F(t, u, v) = -v$. So the conditions $(H_1)$ and $(H_2)$ are satisfied with $L \geqslant 1$. Consequently, (7) and (8) have the unique solution $x(t)$ on $[0, \frac{3}{2}\pi]$ and admit a best approximate solution $f_k^0$ of degree $k$ on $[0, \frac{3}{2}\pi]$, where $k$ is a positive integer. Applying method of steps we get $x(t) = \sin t$, $t \in [0, \frac{3}{2}\pi]$. From Theorem 3 it follows that

$$\lim_{k\to\infty} \sup_{t\in\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]} | f_k^0(t) - \sin t | = 0.$$

In order to obtain $f_k^0(t)$ we proceed as follows :

Here

$$\varphi(t) = \sin t,\ t \in \left[-\frac{\pi}{2}\ 0,\right].$$

Let

$$p_k(t) = c_1 t + c_2 t^2 + \ldots + c_k t^k,\ t \in \left[0, \frac{3\pi}{2}\right]$$

where $c_1, c_2, \ldots, c_k$ are reals. We may note that $c_0 = p_k(0) = \varphi(0) = 0$. So

$$f_k(t) = \begin{cases} \sin t, & t \in \left[-\frac{\pi}{2}, 0\right] \\ p_k(t), & t \in \left[0, \frac{3\pi}{2}\right] \end{cases}$$

and

$$Tf_k(t) = \begin{cases} \sin t, & t \in \left[-\frac{\pi}{2}, 0\right] \\ -\int_0^t f_k\left(s - \frac{\pi}{2}\right) ds, & t \in \left[0, \frac{3\pi}{2}\right]. \end{cases}$$

For

$$t \in \left[0, \frac{\pi}{2}\right],$$

$$T f_k(t) = -\int_{-\frac{\pi}{2}}^{t-\frac{\pi}{2}} f_k(s)\,ds = -\int_{-\frac{\pi}{2}}^{t-\frac{\pi}{2}} \sin s\,ds = \sin t.$$

For

$$t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right],$$

$$T f_k(t) = -\int_{-\frac{\pi}{2}}^{t-\frac{\pi}{2}} f_k(s)\,ds = -\int_{-\frac{\pi}{2}}^{0} f_k(s)\,ds - \int_{0}^{t-\frac{\pi}{2}} f_k(s)\,ds$$

$$= -\int_{-\frac{\pi}{2}}^{0} \sin s\,ds - \int_{0}^{t-\frac{\pi}{2}} p_k(s)\,ds$$

$$= 1 - \left[\frac{c_1}{2}\left(t-\frac{\pi}{2}\right)^2 + \frac{c_2}{3}\left(t-\frac{\pi}{2}\right)^3 + \ldots + \frac{c_k}{k+1}\left(t-\frac{\pi}{2}\right)^{k+1}\right].$$

So

$$\|Sf_k\| = \|f_k - Tf_k\| = \sup_{t \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]} |f_k(t) - Tf_k(t)|$$

$$= \sup_{t \in \left[0, \frac{3\pi}{2}\right]} |f_k(t) - Tf_k(t)|$$

$$= \sup_{t \in \left[0, \frac{3\pi}{2}\right]} |u(t)|,$$

where

$$u(t) = \begin{cases} c_1 t + c_2 t^2 + \ldots + c_k t^k - \sin t, & t \in \left[0, \frac{\pi}{2}\right] \\ c_1 t + \frac{c_1}{2}\left(t-\frac{\pi}{2}\right)^2 + c_1 t^2 + \frac{c_2}{3}\left(t-\frac{\pi}{2}\right)^3 + \ldots + \\ + c_k t^k + \frac{c_k}{k+1}\left(t-\frac{\pi}{2}\right)^{k+1} - 1, & t \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]. \end{cases} \quad \ldots(9)$$

Hence finding a best approximate solution of degree $k$ means to show that

$$\inf_{f_k \in p_k} \|Sf_k\| = \inf_{(c_1, c_2, \cdots, c_k) \in R^k} \sup_{t \in \left[0, \frac{3\pi}{2}\right]} |u(t)|$$

is attained for some vector $\left(c_1^*, c_2^*, \ldots, c_k^*\right) \in R^k$, where $u(t)$ is given by (9).

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