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THE PRODUCT OF TWO FUNCTIONS OF BOUNDED VALUE DISTRIBUTION

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It is known that the product of two functions f and g of bounded value distribution (b.v.d.) is not necessarily of b.v.d. We show that if f and g satisfy appropriate differential equations then fg is of b.v.d. We apply this to show that the product of two Bessel functions of the first kind and integer order is of b.v.d.

INTRODUCTION

Let f be an entire function and let $n\,(a,R,\,1/(f-w))$ denote the number of zeros of $f(z)-w$ $(w\in\mathbb{C})$ in the disc $|\,z-a\,|<R$ $(a\in\mathbb{C},\,R>0)$. We begin with the definitions of functions of bounded value distribution (b.v.d.) and bounded index (b.i.).

Definition 1.1—An entire function f is said to be of b.v.d. if for every $R>0$, there exists a fixed integer $P(R)>0$ such that

$$n\,(a,\,R,\,1/(f-w))\leqslant P\,(R)$$

for all $a\in\mathbb{C}$ and all $w\in\mathbb{C}$.

Definition 1.2—An entire function f is said to be of b.i. if there exists an integer $N>0$ such that for all $z\in\mathbb{C}$ and for all positive k,

$$\max_{0\leqslant j\leqslant N}\left\{\frac{|\,f^{(j)}(z)\,|}{j!}\right\}\geqslant\frac{|\,f^{(k)}(z)\,|}{k!}\,.$$

The concepts are related as the following result of Hayman[7] shows.

Theorem 1.3—An entire function f if of b.v.d. is and only if f' is of b.i.

Lepson[8] defined the concept of b.i. in connection with differential equations. The following theorem of Shah[9] articulates this explicitly.

Theorem 1.4—Every entire solution of the differential equation (DE)

$$w^{(n)}+Q_1(z)\,w^{(n-1)}+\ldots+Q_n(z)\,W=0 \qquad \ldots(1.1)$$

where $Q_i(z)$, $i=1, 2,\ldots, n$, are rational functions bounded at infinity, is of b.i.

For extensions of Theorem 1.4, see Fricke[6] and Shah[10] and the references given there. Combining Theorems 1.3 and 1.4, we have

Corollary 1.5—Every entire solution of the DE (1.1) is of b.v.d..

The concept of b.v.d. is due to Turan[12] and arose in his work on DEs with constant coefficients. In fact, he had obtained Corollary 1.5 above in this situation. For this, see his book : Turan[12] and the references given there.

It is a result of Fricke[4,5] that the product of two functions of b.i. is also of b.i.. However, Fricke[5] also proved that the product of two functions of b.v.d. is not necessarily of b.v.d. . Our aim is to show that if the two functions of b.v.d. satisfy an appropriate DE, then their product is of b.v.d..

2. The Product Of Two Functions

Theorem 2.1—If V and W are entire functions and satisfy DEs $W''-PW=0$ and $V''-QV=0$ where P and Q are rational functions bounded at infinity, then VW satisfies a DE with coefficients which are bounded at infinity and in particular, VW is of b.v.d. .

Proof : Let $y=VW$. Then

$$\begin{aligned} y'' &= V''W+wV\ W'+VW'' \\ &=(Q+P)\,VW-2V'\,W' \\ &=(Q+P)y+2V'\,W' . \end{aligned} \qquad \ldots(2.1)$$

We can rewrite (2.1) as

$$\begin{aligned} \frac{d}{dz}\{y''-(Q+P)y\} &= 2V''\,W'+2V'\,W'' \\ &= 2QW'\,V+2PV'\,W \end{aligned}$$

or

$$y''' - 2(P+Q)y' + (P'+Q')y = -(P+Q)\,(V'W+VW') + 2QW'V + 2PV'W = (P-Q)\,(V'W-VW').$$

The derivative of the last equation gives

$$\begin{aligned} y^{(iv)} - 2(P+Q)y'' - (P'+Q')y' + (P''+Q'')y \\ = (P-Q)(V''W - VW'') \\ = (P-Q)(QVW - PVW) \\ = -(Q-P)^2 y \end{aligned}$$

or

$$y^{(iv)} - 2(P+Q)y'' - (P'+Q')y' + (P''+Q'' + (P-Q)^2)y = 0 . \quad ...(2.2)$$

The derivatives of P and Q vanish at infinity since P and Q are bounded there; thus the coefficients of (2.2) are bounded at infinity. It follows from Corollary (1.5) that $y=VW$ is of b.v.d. .

We now consider an application of this method. The Bessel function $J_n(z)$, where n is an integer, is a solution of the DE

$$W'' + \frac{W'}{z} + \left[1 - \frac{n^2}{z^2}\right] W = 0. \quad ...(2.3)$$

Since $J_n(z)$ is entire and satisfies a DE with coefficients bounded at infinity, therefore $J_n(z)$ is of b.v.d. . We prove:

Theorem 2.2—If n and m are integers, $J_n(z)J_m(z)$ is of b.v.d..

PROOF : Let $J_m = (\theta_m)/\sqrt{z}$. Then (see p. 388 of Rees *et al.*[11])

$$\theta_m'' + \left[1 - \frac{m^2}{z^2} + \frac{1}{4z^2}\right] \theta_m = 0 .$$

Similarly, if $J_n = \theta_n/\sqrt{z}$, then

$$\theta_n'' + \left[1 - \frac{n^2}{z^2} + \frac{1}{4z^2}\right] \theta_n = 0.$$

We cannot, however, use Theorem 2.1 at this point since θ_m and θ_n are not entire. In any case, $\theta_m\,\theta_n$ satisfies a fourth order DE like (2.2) as can be shown by following exactly the same procedure as in Theorem 2.1. Since $\theta_m\,\theta_n = zJ_m(z)\,J_n(z)$, therefore it is entire and $\theta_m\,\theta_n$ is of b.v.d.. Because $z\,J_m\,J_n$ satisfies a DE of the type (2.2), it can be shown by an easy calculation that so does $J_m(z)\,J_n(z)$. Hence $J_m(z)\,J(z)$ is of b.v.d. .

Remarks : The procedure used to find the fourth order DE satisfied by the product of two functions, each of which satisfies a second order DE, is very old. For

example, it was used by Clausen[2] in 1828 to find products of hypergeometric functions which are hypergeometric. Not only is Clausen's method useful in various contexts, but his result itself has an important use in proving a result of Askey and Gasper[1] which turned out to have a crucial application in De Branges[1] proof of the Bieberbach conjecture. See for details De Branges[3]. Clausen himself worked with second order DEs as we have done, but there are possibilities of extending his method to higher order equations.

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