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SECOND-ORDER ELASTICITY FOR A CLASS OF INCOMPRESSIBLE TRANSVERSELY ISOTROPIC ELASTIC MATERIALS

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The equations of Second-Order elasticity are developed in terms of cylindrical polar coordinates for axially symmetric deformations of a class of incompressible transversely isotropic elastic materials. A particular form of strain energy function suggested by Sun and Ueng [*Int. J. Non-linear Mech.* **10** (1975), 129-35] is used. The expressions for Mooney materials are deduced as a particular case.

1. Introduction

Owing to the non-linear character of the equations in the solution of boundary value problems in the theory of finite elasticity, approximate methods are often used. One such method which is widely used is the method of successive approximations. The most comprehensive account of this method is given by Green and Adkins[2] with additional references. In this method it is assumed that the stress and displacement fields admit a power series expansion in a dimensionless parameter ϵ which represents the magnitude of the deformation. The first term in the series corresponds to the usual classical infinitesimal theory of elasticity. Second-order elasticity is obtained by terminating the approximate procedure after second term. Due to the algebraic complexity the higher order approximations are rarely used in practice.

The general theory of second-order elasticity for isotropic materials has been developed by Selvadurai and Spencer[3]. Choi and Shield[4] has discussed the second-order effects in problems for a class of isotropic elastic materials. In this paper an attempt has been made to study the second order effect and develop a general theory for a class of materials which are transversely isotropic and incompressible.

The mathematical equations required for the general theory are given in section 2 and a summary of the finite elasticity theory with axial symmetry expressed in cylindrical polar coordinates is given in section 3. The successive expansions upto second-order are developed in section 4. In section 5 the theory has been applied to a special class of materials with the strain energy function W given by Sun and Ueng[1] and the first and second-order stresses are computed. It is interesting to observe that

the case for second-order effects in Mooney material could be derived as a particular case of the present paper and the results thus obtained agree with the earlier results[2].

2. Mathematical Preliminaries

The basic notations and formulae of finite elasticity used in this paper follow Green and Zerna[5] and Green and Adkins[2].

We fix a rectangular cartesian system of coordinates to locate the particles of an elastic body. The coordinates of a typical particle in the undeformed body are denoted by x_i and the coordinates of the particle in the deformed body are denoted by y_i. We also define a general curvilinear coordinate system θ^i $(= \theta_i)$ so that

$$x_i = x_i(\theta_1, \theta_2, \theta_3) \qquad \ldots(2.1)$$

where $x_i(\theta_1, \theta_2, \theta_3)$ is a single valued function with non-zero Jacobian and possesses continuous derivatives upto any required order. The system θ_i moves with the body as it deforms so that deformation of the body is defined by

$$y_i = y_i(\theta_1, \theta_2, \theta_3). \qquad \ldots(2.2)$$

The displacement vector $\mathbf{u}(\theta_1, \theta_2, \theta_3)$ may be expressed as

$$\mathbf{u} = u^i \mathbf{g}_i = u_j \mathbf{g}^j \qquad \ldots(2.3)$$

where $\mathbf{g}_i$, $\mathbf{g}^j$ are the covariant and contravariant base vectors for the θ_i system in the undeformed body and repeated indices show summation.

The strain tensor γ_{ij} defined by

$$\gamma_{ij} = 1/2\ (G_{ij} - g_{ij}) \qquad \ldots(2.4)$$

where g_{ij} and G_{ij} are the metric tensors for the θ_i systems in the undeformed and deformed bodies respectively.

For transversely isotropic materials in which there is symmetry about the x_3-axis the appropriate form of strain energy function W can be expressed as[2]

$$W = W(I_1, I_2, I_3, K_1, K_2) \qquad \ldots(2.5)$$

where I_1, I_2, I_3, K_1 and K_2 are the strain invariants are given by

$$I_1 = 3 + 2\gamma^r_r \quad = g^{rs} G_{rs}$$

$$I_2 = 3 + 4\gamma^r_r + 2\left(\gamma^r_r\ \gamma^s_s - {}^{s}_{\gamma r}\ \gamma^r_s\right) = g_{rs}\, G^{rs}\, I_3$$

$$I_3 = \left| \delta^r_s + 2\gamma^r_s \right| = G/g$$

$$K_1 = e_{33}$$

$$K_2 = e_{31}\, e_{31} + e_{32}\, e_{32} \qquad \ldots(2.6)$$

and where

$$e_{rs} = \frac{\partial \theta^i}{\partial x^r} \frac{\partial \theta^j}{\partial x^s} \gamma_{ij}, \quad \gamma_{ij} = g^{ik}\, \gamma_{kj}. \qquad \ldots(2.7)$$

The state of stress at a point in the deformed body is determined by the symmetric stress tensor s^{ij} measured per unit area of the undeformed body and refered to base vector $\mathbf{G}_i$.

The stress tensor s^{ij} for transversely isotropic material is given by

$$s^{ij} = 2g^{ij} \frac{\partial W}{\partial I_1} + 2B^{ij} \frac{\partial W}{\partial I_2} + 2G^{ij} \frac{\partial W}{\partial I_3} + M^{ij} \frac{\partial W}{\partial K_1} + N^{ij} \frac{\partial W}{\partial K_2} \qquad \ldots(2.8)$$

where

$$B^{ij} = I_1\, g^{ij} - g^{ir}\, g^{js}\, G_{rs} \qquad \ldots(2.9)$$

$$M^{ij} = \frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^j}{\partial x^3} \qquad \ldots(2.10)$$

and

$$N^{ij} = \left(\frac{\partial \theta^i}{\partial x^3} \frac{\partial \theta^i}{\partial x^\alpha} + \frac{\partial \theta^i}{\partial x^\alpha} \frac{\partial \theta^j}{\partial x^3} \right) e_{\alpha 3} \qquad \ldots(2.11)$$

where α takes values 1, 2.

It is some times more convenient to work in terms of the non symmetric stress tensor t^{ij} measured per unit area of the undeformed body and refered to base vectors $\mathbf{g}_i$. Then we have

$$t^{ij} = s^{ir} \left(\delta^j_r + u^j \mid_r \right) \qquad \ldots(2.12)$$

where a single line indicates covariant differentiation with the respect to θ_i using the metric tensor g_{ij} and g^{ij}.

The equation of equilibrium are given by

$$t^{ij} \mid_j + \rho_0 ({}_0F^i) = 0 \qquad \ldots(2.13)$$

where ρ_0 is the density of the undeformed body and ${}_0F^i$ is component of body force vector.

3. Finite Elasticity with Axial Symmetry

We choose the moving system of coordinates θ_i so that

$$(\theta_1, \theta_2, \theta_3) = (r, \theta, z) \qquad \ldots(3.1)$$

where (r, θ, z) are cylindrical polar coordinates in the undeformed body,

$$x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = z. \qquad \ldots(3.2)$$

If (R, Θ, Z) are cylindrical polar coordinates in the deformed body so that

$$y_1 = R\cos\Theta,\ y_2 = R\sin\Theta,\ y_3 = Z.$$

Then we have for axially symmetric deformations

$$R = R(r, z),\ \Theta = \theta,\ Z = Z(r, z) \qquad \ldots(3.4)$$

and the components displacement vector are

$$u^1 = R - r,\ u^2 = 0,\ u^3 = Z - z. \qquad \ldots(3.5)$$

From (3.1) and (3.2) we have

$$\left[\frac{\partial x^s}{\partial\theta^s}\right] = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \ldots(3.6)$$

and

$$\left[\frac{\partial\theta^r}{\partial x^s}\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \ldots(3.7)$$

From (3.6) and (3.7) the metric tensors are given by

$$g_{ij} = \text{diag}\ [1, r^2, 1]\ g^{ij} = \text{diag}\,[1, 1/r^2, 1]. \qquad \ldots(3.8)$$

Also (3.3) and (3.4) give

$$G_{ij} = \begin{Bmatrix} R_{,r}^2 + Z_{,r}^2 & 0 & R_{,r}R_{,z} + Z_{,r}Z_{,z} \\ 0 & R^2 & 0 \\ R_{,r}R_{,z} + Z_{,r}Z_z & 0 & R_{,z}^2 + Z_{,z}^2 \end{Bmatrix} \qquad \ldots(3.9)$$

$$G^{ij} = \begin{bmatrix} \left(R_{,z}^2 + Z_{,z}^2 \right)/\Delta^2 & 0 & -(R_{,r} R_{,z} + Z_{,r} Z_{,z})/\Delta^2 \\ 0 & 1/R_2 & 0 \\ -(R_{,r} R_{,z} + Z_{,r} Z_{,z})/\Delta^2 & 0 & \left(R_{,r}^2 \; r Z_{,r}^2 \right)/\Delta^2 \end{bmatrix} \quad \ldots(3.10)$$

where

$$\Delta = \frac{\partial (R, Z)}{\partial (r, z)} = R_{,r} Z_{,z} - R_{,z} Z_{,z}. \quad \ldots(3.11)$$

From (2.4) (3.8) and (3.9) we have

$$2\gamma_{ij} = \begin{bmatrix} R_{,r}^2 + Z_{,r}^2 - 1 & 0 & R_{,r} R_{,z} + Z_{,r} Z_{,z} \\ 0 & R^2 - r^2 & 0 \\ R_{,r} R_{,z} + Z_{,r} Z_{,z} & 0 & R_{,z}^2 + Z_{,z}^2 - 1 \end{bmatrix}. \quad \ldots(3.12)$$

Substituting (3.8) — (3.12) in (2.6) and using (2.7) the strain invariants are found to be

$$I_1 = R_{,r}^2 + R_{,z}^2 + Z_{,r}^2 + Z_{,z}^2 + R^2/r^2 \quad \ldots(3.13)$$

$$I_2 = R^2 \left(R_{,r}^2 + R_{,z}^2 + Z_{,r}^2 + Z_{,z}^2 \right)/r^2 + \Delta^2 \quad \ldots(3.14)$$

$$I_3 = R^2 (R_{,r} Z_{,z} - R_{,z} Z_{,z})^2/r^2 \quad \ldots(3.15)$$

$$K_1 = 1/2 \left(R_{,z}^2 + Z_{,z}^2 - 1 \right) \quad \ldots(3.16)$$

$$K_2 = 1/4 \; (R_{,r} R_{,z} + Z_{,r} Z_{,z})^2. \quad \ldots(3.17)$$

From (2.9) the tensor B^{ij} is given by

$$B^{ij} = \begin{bmatrix} R_{,z}^2 + Z_{,z}^2 + R^2/r^2 & 0 & -(R_{,r} R_{,z} + Z_{,r} Z_{,z}) \\ 0 & \left(R_{,r}^2 + R_{,z}^2 + Z_{,r}^2 + Z_{,z}^2 \right)/r^2 & 0 \\ -(R_{,r} R_{,z} + Z_{,r} Z_{,z}) & 0 & R_{,r}^2 + Z_{,r}^2 + R^2/r^2 \end{bmatrix} \quad \ldots(3.18)$$

From (2.7) and (3.7) the tensors $M^{(i)}$ and $N^{(i)}$ in (2.11) have components

$$M^{33} = 1, \; N^{13} = 1/2 \, (R_{,r} \, R_{,z} + Z_{,r} \, Z_{,z}) \qquad \ldots(3.19)$$

and the remaining components become zero.

4. Second-order Elasticity : Incompressible Case

We now assume that the displacement vector can be expanded as an asymptotic power series in a dimensionless parameter ϵ which represents the magnitude of the deformation. Second order elasticity results from retaining the terms involving ϵ and ϵ^2 and neglecting the higher powers of ϵ.

Thus we write from (3.5)

$$u^1 = \epsilon \, v^1 \, (r, z) + \epsilon^2 \, w^1 \, (r, z) + \ldots \qquad \ldots(4.1)$$

$$u^3 = \epsilon \, v^3 \, (r, z) + \epsilon^2 w^3 \, (r, z) + \ldots$$

Equations (3.5) and (4.1) with (3.13) – (3.17) give

$$I_1 = 3 + 2\epsilon \left(v^1_{,r} + v^1/r + v^3_{,z} \right) + \epsilon^2 \left[2w^1_{,r} + 2w^1/r + 2w^3_{,z} \right.$$

$$\left. + \left(v^1_{,r} \right)^2 + \left(v^1_{,z} \right)^2 + \left(v^3_{,r} \right)^2 + \left(v^3_{,z} \right)^2 + (v^1/r)^2 \right] + \ldots \qquad \ldots(4.2)$$

$$I_2 = 3 + 4 \, \epsilon \left(v^1_{,r} + v^1/r + v^3_{,z} \right) + \epsilon^2 \left[4w^1_{,r} + 4w^1/r + 4w^3_{,z} \right.$$

$$+ 2 \left(v^1_{,r} + v^3_{,z} \right)^2 + \left(v^1_{,z} - v^3_{,r} \right)^2 + 2 \, (v^2/r)^2$$

$$\left. + 4 \left(v^1_{,r} + v^3_{,z} \right) v^1/r \right] + \ldots \qquad \ldots(4.3)$$

$$I_3 = 1 + 2\epsilon \left(v^1_{,r} + v^1/r + v^3_{,z} \right) + \epsilon^2 \left[2w^1_{,r} + 2w^1/r + 2w^3_{,z} \right.$$

$$+ 4v^1_{,r} \, v^3_{,z} - 2v^1_{,z} \, v^3_{,r} + \left(v^1_{,r} \right)^2 + \left(v^3_{,z} \right)^2 + (v^1/r)^2$$

$$\left. + 4 \left(v^1_{,r} + v^3_{,z} \right) v^1/r \right] + \ldots \qquad \ldots(4.4)$$

$$K_1 = \epsilon \, v^3_{,z} + 1/2 \, \epsilon^2 \left[2w^3_{,z} + \left(v^1_{,z} \right)^2 \left(v^3_{,z} \right)^2 \right] + \ldots \qquad \ldots(4.5)$$

$$K_2 = 1/4\ \epsilon^2 \left(v^1_{,z} + v^3_{,r} \right)^2 + \ldots \qquad \ldots(4.6)$$

When the body is incompressible, we have $I_3 = 1$ and hence from (4.4) we get the first and second-order incompressibility conditions as

$$v^1_{,r} + v^1/r + v^3_{,z} = 0 \qquad \ldots(4.7)$$

$$w^1_{,r} + w^1/r + w^3_{,z} = 1/2 \left[\left(v^1_{,r} \right)^2 + (v^1/r)^2 + \left(v^3_{,z} \right)^2 + 2\, v^1_{,z}\, v^3_{,r} \right] \qquad \ldots(4.8)$$

From (4.7) and (4.8), (4.2) and (4.3) reduce to

$$I_1 = 3 + \epsilon^2 \left[2 \left(v^1_{,r} \right)^2 + 2 (v^1/r)^2 + 2 \left(v^3_{,z} \right)^2 + 2 \left(v^1_{,z} + v^3_{,z} \right)^2 \right] + \ldots \qquad \ldots(4.9)$$

$$I_2 = 3 + \epsilon^2 \left[2 \left(v^1_{,r} \right)^2 + (v^1/r)^2 + 2 \left(v^3_{,z} \right)^2 + \left(v^1_{,z} + v^3_{,r} \right)^2 \right] + \ldots \qquad \ldots(4.10)$$

The components of metric tensor B^{ij} in (3.8) with (3.5) and (4.1) and using (4.7) and (4.8) can be expanded as

$$B^{11} = 2 - 2\epsilon \left(v^1_{,r} \right) + \epsilon^2 \left[- 2w^1_{,r} + (v^1/r)^2 + \left(v^3_{,z} \right)^2 - \left(v^3_{,r} \right)^2 + \left(v^1_{,z} + v^3_{,r} \right)^2 \right] + \ldots \qquad \ldots(4.11)$$

$$r^2 B^{22} = 2 - 2\epsilon\, (v^1/r) + \epsilon^2 \left[- 2w^1/r + 2 \left(v^1_{,r} \right)^2 + (v^1/r)^2 + 2 \left(v^3_{,z} \right)^2 + \left(v^1_{,z} + v^3_{,r} \right)^2 \right] + \ldots \qquad \ldots(4.12)$$

$$B^{33} = 2 - 2\epsilon \left(v^3_{,z} \right) + \epsilon^2 \left[- 2w^3_{,z} + 2\, (v^1/r)^2 + \left(v^1_{,r} \right)^2 - \left(v^1_{,z} \right)^2 - \left(v^1_{,z} + v^3_{,z} \right)^2 \right] + \ldots \qquad \ldots(4.13)$$

$$B^{13} = - \epsilon \left(v^1_{,z} + v^2_{,z} \right) - \epsilon^2 \left[w^1_{,z} + w^3_{,r} + v^1_{,r}\, v^1_{,z} + v^3_{,r}\, v^3_{,z} \right] + \ldots \qquad \ldots(4.14)$$

We also need the expansions of the components of metric tensor G^{ij}. From (3.5), (3.10), (3.11), (4.1), (4.7) and (4.8) we have

$$G^{11} = 1 - 2\epsilon\, v^1_{,r} + \epsilon^2 \left[2w^3_{,z} + 2w^1/r + \left(v^1_{,r} \right)^2 + \left(v^1_{,z} \right)^2 + \left(v^3_{,z} \right)^2 + \left(4v^3_{,z}\, v^1/r \right) \right] + \ldots \qquad \ldots(4.15)$$

$$r^2 G^{22} = 1 - 2\epsilon\, v^1/r - \epsilon^2 \left[2w^1/r - 3\,(v^1/r)^2 \right] + \ldots \qquad \ldots (4.16)$$

$$G^{33} = 1 - 2\epsilon\, v^3_{,z} + \epsilon^2 \left[2w^1_{,r} + 2w^1/r + \epsilon\,(v^1_{,r})^2 + (v^1/r)^2 + \left(v^3_{,r} \right)^2 + 4v^1_{,r}\, v^1/r \right] + \ldots \qquad \ldots(4.17)$$

$$G^{13} = -\epsilon \left(v^1_{,z} + v^3_{,r} \right) - \epsilon^2 \left[w^1_{,z} + w^3_{,r} + v^1_{,z}\, v^1_{,r} + v^3_{,z}\, v^3_{,r} + 2 \left(v^1_{,z} + v^3_{,r} \right) v^1/r \right] + \ldots \qquad \ldots(4.18)$$

5. Stress-Strain Relations

The stress-strain relations for incompressible transversely isotropic materials can be expressed, as in[2]

$$s^{ij} = 2g^{ij} \frac{\partial W}{\partial I_1} + 2B^{ij} \frac{\partial W}{\partial I_2} + M^{ij} \frac{\partial W}{\partial K_1} + N^{ij} \frac{\partial W}{\partial K_2} + pG^{ij} \qquad \ldots(5.1)$$

where p is a scalar function of the coordinates θ_i.

We take an explicit form of the strain energy function W given in Sun and Ueng[1]

$$W = A \,|\, (I_1 - 3) + \Gamma_1 (I_2 - 3) + \Gamma_2 K_1 + 3 K_2 + \Gamma_4 K_1^2 + \Gamma_5 (I_1 - 3) K_1^2 + \Gamma_6 (I_1 - 3) K_2 \,| \qquad \ldots (5.2)$$

where A, $\Gamma_1, \ldots \Gamma_6$ are material constants. These material constants need to be determined experimentally. By appropriate choice of these constants (5.2) represents the strain energy function for this particular kind of material.

We further assume[3] that the stress tensor t^{ij} and the scalar function p can be expressed as power series in ϵ, in the forms

$$t^{ij} = t_0^{ij} + \epsilon\, t_1^{ij} + \epsilon^2\, t_2^{ij} + \ldots \qquad \ldots(5.3)$$

$$p = p_0 + \epsilon\, p_1 + \epsilon^2\, p_2 + \ldots \qquad \ldots(5.4)$$

If we substitute (5.2) and (5.4) in to (5.1) ang using (2.12) with the help of the results of the previous sections, the terms independent of ϵ in (5.3) are given by

$$t_0^{11} = 2A + 4A\,\Gamma_1 + p_0 = r^2 t_0^{22}\,,\quad t_0^{33} = 2A + 4A\,\Gamma_1 + A\,\Gamma 2 + p_0. \qquad \ldots(5.5)$$

If t^{ij} is to satisfy the equilibrium equations (2.13,) t_0^{ij} must be a uniform hydrostatic pressure. Since such stress is, for incompressible material, independent of the deformation, we may without loss of generality take it to be zero, and if necessary superpose any required constant hydrostatic stress on any stress field under this assumption. Thus from (5.5) we take

$$p_0 = -(2A + 4A\,\Gamma_1) \text{ and } \Gamma_2 = 0. \qquad \ldots(5.6)$$

With the help of (5.6) the stresses corresponding to ϵ are

$$t_1^{11} = p_1 + 4(A + A\,\Gamma_1)\, v^1_{,r}$$

$$r^2 t_1^{22} = p_1 + 4(A + A\,\Gamma_1)\, v^1/r$$

$$t_2^{33} = p_1 + 2(2A + 2A\,\Gamma_1 + A\,\Gamma_4)\, v^3_{,z}$$

$$t_1^{31} = t_1^{13} = (2A + 2A\,\Gamma_1/+A\Gamma_{3/2})\left(v^1_{,z} + v^3_{,r}\right). \qquad \ldots(5.7)$$

The stresses corresponding to ϵ^2 are given by

$$t_2^{11} = p_2 - p_1 v^1_{,r} - 4A\Gamma_1 w^1_{,r} - 4(A+A_1)\left(w^3_{,z} + w^1/r\right) + 2(A + A\,\Gamma_1)$$

$$\left[2\left(v^1_{,r}\right)^2 - (v^1/r)^2 - \left(v^1_{,z}\right)^2\right] - 2A\Gamma_1\left(v^3_{,r}\right)^2$$

$$+ 2(A\,\Gamma_5 - A - A\,\Gamma_1)$$

(equation continued on p. 703)

$$\left(v^3_{,z} \right)^2 - 8\,(A + 2A\,\Gamma_1)\, v^3_{,z}\, v^1/r + (A\,\Gamma_6/2 + 2A\,\Gamma_1)$$

$$\times \left(v^1_{,z} + v^3_{,r} \right)^2 + (2A + 2A\,\Gamma_1 + A\,\Gamma_3/2)\left(v^1_{,z} + v^3_{,r} \right) v^1_{,z} \qquad \ldots(5.8)$$

$$r^2 t^{22}_2 = p_2 - p_1 v^1/r + 4\,(A + A\,\Gamma_1)\, w_1/r + 4A\,\Gamma_1 \left(v^1_{,r} \right)^2$$

$$- 2\,(A + 3A\,\Gamma_1)\,(v^1/r)^2 + 2\,(2A\,\Gamma_1 + A\,\Gamma_5)\left(v^3_{,z} \right)^2$$

$$+ (2A\,\Gamma_1 + A\,\Gamma_6/2)\left(v^1_{,z} + v^3_{,r} \right)^2 \qquad \ldots(5.9)$$

$$t^{33}_2 = p_2 - p_1 v^3_{,z} - 4(A + 2A\,\Gamma_1)\left(w^1_{,r} + w^1/r \right) + 2\,(A\,\Gamma_4 - 2A\,\Gamma_1)$$

$$w^3_{,z} - 2\,(A + A_1)$$

$$\left(v^1_{,r} \right)^2 - 2A\,(v^1/r)^2 - (2A\,\Gamma_1 - A\,\Gamma_4)\left(v^1_{,z} \right)^2 - 2\,(A + 2A\Gamma_1)$$

$$\times \left(v^3_{,r} \right)^2 + (4A + 4A\,\Gamma_1 + 5A\,\Gamma_4 + 2A\,\Gamma_5)\left(v^3_{,z} \right)^2$$

$$- 2\,(A_1 - A\Gamma_6/2)\left(v^1_{,z} + v^3_{,r} \right)^2 - 8\,(A + 2A\,\Gamma_5)\, v^1_{,r}\, v^1/r$$

$$+ (2A + 2A\,\Gamma_1 + A\,\Gamma_3/2)\left(v^1_{,z} + v^3_{,r} \right) v^3_{,r} \qquad \ldots(5.10)$$

$$t^{13}_2 = - p_1 v^1_{,z} - (2A + 2A\,\Gamma_1 + A\,\Gamma_3/2)\left(w^1_{,z} + w^3_{,r} + v^1_{,r}\, v^1_{,z} \right.$$

$$\left. + v^3_{,r}\, v^3_{,z} \right) + 4\,(A + 2A\,\Gamma_1)\left(v^1_{,z} + v^3_{,r} \right) v^1/r + 4\,(A + A\,\Gamma_1)$$

$$\times\, v^1_{,r}\, v^3_{,z} + (2A + 2A\,\Gamma_1 + A\,\Gamma_3/2)\left(v^1_{,z} + v^3_{,r} \right) v^3_{,z}$$

$$\ldots(5.11)$$

$$t^{31}_2 = - p_1 v^3_{,r} - (2A + 2A\,\Gamma_1 + A\,\Gamma_3/2)\left(w^1_{,z} + w^3_{,r} + v^1_{,r}\, v^1_{,z} \right.$$

(equation continued on p. 704)

$$+ v^3_{,r}\ v^3_{,z}\ \Big) + 4\,(A + 2A\,\Gamma_1)\Big(\ v^1_{,z} + v^3_{,r}\ \Big)\,v^1/r + (2A + 2A\,\Gamma_1$$

$$+ A\,\Gamma_3/2)\Big(\ v^1_{,z} + v^3_{,r}\ \Big)\,v^1_{,r}\ . \qquad \ldots(5.12)$$

Substituting (5.3) in to (2.13) we have the equation of equilibrium without body forces are

$$t^{11}_{1,r} + t^{31}_{1,z} + \Big(t^{11}_1 - r^2 t^{22}_1\ \Big)/r = 0$$

$$t^{13}_{1,r} + t^{13}_1/r + t^{33}_{1,z} = 0. \qquad \ldots(5.13)$$

The second order stress t^{ij}_{α} satisfy the same equations (5.13) as of first order stresses.

It is interesting to observe that the case of second-order effects in Mooney materials becomes a particular case if we take

$$A = C_1,\ A_1 = C_2 \text{ and } \Gamma_2 = \Gamma_3 = \ldots = \Gamma_6 = 0 \qquad \ldots(5.14)$$

and the results thus derived tally with the earlier results[2].

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