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SPHERICAL CAVITY IN A SEMI-INFINITE SOLID WITH ITS FREE SURFACE RIGIDLY CLAMPED

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The present paper contains an approach to deal with distribution of stresses and deformation in a stressed semi-infinite solid containing a spherical cavity and the free surface of the solid is rigidly clamped. The problem can be reduced to one of two simply connected domains, namely a semi-infinite region and an infinite region excluding the cavity. The two stress functions in each region are expressed in cylindrical and spherical harmonics. The prescribed boundary conditions on the free surface of semi-infinite body and on the surface of the cavity are satisfied with the relation between cylindrical and spherical harmonics. Numerical results are given for quantities of physical interest by taking different radii of the cavity and the distance of the centre of cavity from the free surface is assumed as the unit of measurement.

1. INTRODUCTION

We have recently given an analysis of the mixed boundary value problems[1-3] of embedded solids containing different types of inclusions. Generally the problems of semi-infinite solid with different types of inclusions have been solved, when the free boundary of the solid is stress free. But in most of the engineering problems the free surface of the solids has to be kept rigidly clamped to bear over all distribution of stresses and strains. Keeping this in view the present paper deals with the problem of spherical cavity in a semi-infinite solid with the free boundary rigidly clamped so that we can see the effect of the distribution of stresses and displacements in this case. The problem of a semi-infinite body containing a cavity constitutes a plane hydrostatic pressure parallel to the z plane if there is a traction at infinity. The method adopted here is the division of elastic solid into two sub regions; one is the semi-infinite region

with cavity while the other is infinite region excluding cavity. The boundary conditions on the surface of the cavity and on the free surface of semi-infinite solid are satisfied by the relations between two harmonic functions. This method is, however, useful to those problems which are solved by using spherical dipolar coordinates. In the present case spherical dipolar coordinates system can not be used so the problem has been solved by assuming the Boussineq's stress function in both the regions. The quantities of physical interest are obtained for $a = 0.2, 0.4, 0\ 6, 0.7, 0.8, 0.9$ where a is radius of the cavity and the unit of measurement is taken as the distance of the free surface of the solid from the centre of the cavity.

2. Formulation of the Problem

We consider a semi-infinite, isotropic, homogeneous elastic solid containing a spherical cavity whose centre 0 is taken as the origin. The cylindrical and spherical co-ordinates of a typical point are denoted by (r, θ, z) and $R, (\theta, \psi)$ as shown in Fig. 1. They are related to each other as $R \sin \psi = r$, $R \cos \psi = z$.

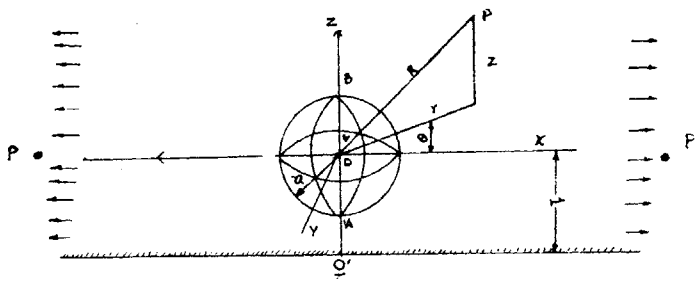

Fig. 1. Coordinate system (r,θ,Z) and (R,θ,ψ).

The free surface of the solid is given by $z = -1$, the half space by $z \geqslant -1$ and the sperical cavity by $R = a$.

In case of axial symmetry, the displacement vector assumes the form $(u, 0, w)$. Following Papkovich and Boussinesq the general solution of the displacement equation of equilibrium, in case of torsion free rotational symmetry and in the absence of body force, is given by the two displacement fields according to cylindrical coordinates[7].

$$2G\,u = \frac{\partial \varphi_0}{\partial r} + z\frac{\partial \varphi_3}{\partial r},\ v = 0 \qquad \ldots(1)$$

$$2G\,w = \frac{\partial \varphi_0}{\partial z} + z\frac{\partial \varphi_3}{\partial z} - (3 - 4\eta)\,\varphi_3 \qquad \ldots(2)$$

where

$$\nabla^2 \varphi_0 = \nabla^2 \varphi_3 = 0, \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

and (u, v, w) denote the radial, transverse and axial components of displacement vector while G and η are shear modulus and Poisson's ratio respectively.

The above stress functions φ_0 and φ_3 give the following stresses in cylindrical coordinates

$$\left.\begin{aligned}
\sigma_r &= -\frac{1}{r}\frac{\partial \varphi_0}{\partial r} - \frac{\partial^2 \varphi_0}{\partial z^2} - \frac{z}{r}\frac{\partial \varphi_3}{\partial r} - z\frac{\partial^2 \varphi_3}{\partial z^2} - 2\eta\frac{\partial \varphi_3}{\partial z},\\
\sigma_\theta &= \frac{1}{r}\frac{\partial \varphi_0}{\partial r} + \frac{z}{r}\frac{\partial \varphi_3}{\partial r} - 2\eta\frac{\partial_3 \varphi}{\partial z},\\
\sigma_z &= \frac{\partial^2 \varphi_0}{\partial z^2} - 2(1-\eta)\frac{\partial \varphi_3}{\partial z} + z\frac{\partial^2 \varphi_3}{\partial z^2}, \quad \tau_{rz} = \frac{\partial^2 \varphi_0}{\partial r \partial z}\\
&\quad - (1-2\eta)\frac{\partial \varphi_3}{\partial r} + z\frac{\partial^2 \varphi_3}{\partial r \varphi z},\\
\tau_{\theta z} &= \tau_{r\theta} = 0.
\end{aligned}\right\} \quad \ldots(3)$$

The above equations on transformation into spherical coordinates give the following expressions for stress components in spherical coordinates

$$\sigma_R = \frac{\partial^2 \varphi_0}{\partial R^2} + R\cos\psi\frac{\partial^2 \varphi_3}{\partial R^2} - 2(1-\eta)\cos\psi\frac{\partial \varphi_3}{\partial R} + 2\eta\frac{\partial \varphi_3}{\partial \psi},$$

$$\sigma_R = \frac{1}{R}\frac{\partial \varphi_0}{\partial R} + \frac{\cot\psi}{R^2}\frac{\partial \varphi_0}{\partial \psi} - (1-2\eta)\cos\psi\frac{\partial \varphi_3}{\partial R} + \frac{1}{R} \times [\operatorname{cosec}\psi - (1-2\eta)\sin\psi]\frac{\partial \varphi_3}{\partial \psi},$$

$$\sigma_\Psi = \frac{1}{R^2}\frac{\partial^2 \varphi_0}{\partial \psi^2} + \frac{1}{R}\frac{\partial \varphi_0}{\partial R} + (1-2\eta)\cos\psi\frac{\partial \varphi_3}{\partial R} + \frac{\cos\psi}{R}\frac{\partial^2 \varphi_3}{\partial \psi^2} + 2(1-\eta)\frac{\sin\psi}{R}\frac{\partial \varphi_3}{\partial \psi},$$

$$\tau_{R\Psi} = \frac{1}{R}\frac{\partial^2 \varphi_0}{\partial R \partial \psi} - \frac{1}{R^2}\frac{\partial \varphi_0}{\partial \psi} + (1-2\eta)\sin\psi\frac{\partial \varphi_3}{\partial R} + \cos\psi\frac{\partial^2 \varphi_3}{\partial R \partial \psi} - 2(1-\eta)\frac{\cos\psi}{R}\frac{\partial \varphi_3}{\partial \psi},$$

$$\tau_{R\theta} = \tau_{\theta\Psi} = 0. \quad \ldots(4)$$

Suppose that the free surface of the solid $z = -1$ is rigidly clamped, the boundary conditions to be satisfied are

$$2G\left(\frac{w}{p}\right) z = -1 = 2G\left(\frac{u}{p}\right) z = -1 = 0, r \geqslant 0 \qquad \ldots(5)$$

and on the surface of the cavity

$$\left(\frac{\sigma_R}{p}\right) R = a = 0 \qquad \ldots(6)$$

$$\left(\frac{\tau_{R\Psi}}{p}\right) R = a = 0. \qquad \ldots(7)$$

In order to satisfy the boundary conditions (5)-(7), we assume φ_0 and φ_3 as follows

$$[I] \quad \begin{cases} \varphi_0 = AR^2 P_2(\mu) & \ldots(8) \\ \varphi_3 = BR P_1(\mu) & \ldots(9) \end{cases}$$

where

$$A = -\frac{1-\eta}{1+\eta} p \quad B = -\frac{p}{1+\eta} \qquad \ldots(10)$$

and $P_m(\mu)$ is the Legendre's function of the first kind of order m and μ is cos ψ.

Substituting $[I]$ into eqns. (3) and (4) the conditions (5)-(7) are not satisfied but the following stresses and displacement fields are developed and the surface of the cavity is not stress free

$$\sigma_z = \tau_{rz} = 0, \frac{\sigma_R}{p} = \frac{2}{3}[P_0(\mu) - P_2(\mu)], \frac{\tau_{R\Psi}}{p} = \frac{1}{3} \sin\psi P'_2(\mu) \qquad \ldots(11)$$

$$2G\frac{w}{p} = -\frac{2\eta z}{1+\eta}, 2G\frac{u}{p} = \frac{1-\eta}{1+\eta} r. \qquad \ldots(12)$$

Equations (11) and (12) for $z = -1$ and $R = a$ become

$$2G\left(\frac{w}{p}\right) z = -1 = \frac{2\eta}{1+\eta}, r > 0 \qquad \ldots(13)$$

$$2G\left(\frac{u}{p}\right) z = -1 = \frac{1-\eta}{1+\eta} r, r \geqslant 0 \qquad \ldots(14)$$

$$\left(\frac{\sigma_R}{p}\right) R = a = \frac{2}{3}[P_0(\mu) - P_2(\mu)] \qquad \ldots(15)$$

$$\left(\frac{\tau_{R\psi}}{p}\right) R = a = \frac{1}{3} \sin \psi \; P_2' \; (\mu) \qquad \ldots(16)$$

and at infinity (∞) all the stresses and displacements derived vanish.

Keeping the conditions (13) to (16) in veiw the solution of this problem [with the boundary conditions (5) to (7)] will be acquired if we get the solution which satisfies the following boundary conditions :

(i) $$2G \left(\frac{w}{p}\right) z = -1 = 1 = -\frac{2\eta}{1+\eta}, \; r \geqslant 0 \qquad \ldots(17)$$

$$2G \left(\frac{u}{p}\right) z = 1 = -\frac{1-\eta}{1+\eta} r, \; r \geqslant 0. \qquad \ldots(18)$$

(ii) On the surface of cavity

$$\left(\frac{\sigma_R}{p}\right) R = a = -\frac{2}{3} [P_0 (\mu) - P_2 (\mu)] \qquad \ldots(19)$$

$$\left(\frac{\tau_{R\psi}}{p}\right) R = a = -\frac{1}{3} \sin \psi \; P_2' \; (\mu). \qquad \ldots(20)$$

For the solution of the present problems as mentioned before we make use of two groups of stress functions which have no singularities in the semi-infinite body except at the origin. These functions are expressed by cylindrical and spherical harmonics

[II]
$$\varphi_0 = p \sum_{m=0}^{\infty} \frac{A_m P_m (\mu)}{R^{m+1}}, \qquad \ldots(21)$$

$$\varphi_3 = p \sum_{m=0}^{\infty} B_m \frac{P_m (\mu)}{R^{m+1}} \qquad \ldots(22)$$

[III]
$$\varphi_0 = p \int_0^{\infty} \varphi_1 (\lambda) \, e^{-\lambda z} J_0 (\lambda r) \, d\lambda \qquad \ldots(23)$$

$$\varphi_3 = p \int_0^{\infty} \lambda \varphi_2 (\lambda) \, e^{-\lambda z} J_0 (\lambda r) \, d\lambda \qquad \ldots(24)$$

where $J_0 (\lambda r)$ denotes Bessel functions of first kind and Am, Bm are unknown constants while $\varphi_1 (\lambda)$, $\varphi_2 (\lambda)$ are arbitrary functions of λ. With those fuctions we get the components of displacement vector and stress tensor through (1)-(4) and making use of the linear combinations of [I], [II] and [III].

3. Method of Solution

The stress functions have to be transformed into one kind of coordinates so as to fulfill the boundary conditions both on the free boundary and on the surface of cavity. Following Sneddon[4] it is found that for $z \geqslant 0$

$$\frac{P_n(\mu)}{R^{n+1}} = \frac{1}{n!}\int_0^\infty \lambda^n e^{-\lambda z} J_0(\lambda r)\, dr \quad z \geqslant 0. \qquad \ldots(25)$$

For $z \geqslant 0$ we have to replace z by $(-z)$ and ψ by $(\pi - \psi)$ so that from (25) we get

$$\frac{P_n(\mu)}{R^{n+1}} = \frac{(-1)^n}{n!}\int_0^\infty \lambda^n e^{\lambda z} J_0(\lambda r)\, d\lambda \quad z \leqslant 0. \qquad \ldots(26)$$

With the above relation the function [II] are changed as

[IV]
$$\varphi_0 = p \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} A^m \int_0^\infty \lambda^m e^{\lambda z} J_0(\lambda r)\, d\lambda \qquad \ldots(27)$$

$$\varphi_3 = p \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} B_m \int_0^\infty \lambda^m e^{\lambda z} J_0(\lambda r)\, d\lambda. \qquad \ldots(28)$$

Substituting equations (23), (24), (27) and (28) into (1) and (2) and satisfying the boundary conditions (17) and (18) we get

$$\lambda \varphi_1(\lambda) e^{\lambda} + \lambda \varphi_2(\lambda)(3 - 4\eta - \lambda) e^{\lambda} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \lambda^m$$

$$\{\lambda A_m - B_m(3 - 4\eta + \lambda)\} e^{-\lambda} \qquad \ldots(29)$$

$$\varphi_1(\lambda) e^{\lambda} - \lambda \varphi_2(\lambda) e^{\lambda} = -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \lambda^m (A_m - B_m) e^{-\lambda}. \qquad \ldots(30)$$

Solving the above equations we get

$$(3 - 4\eta)\lambda \varphi_2(\lambda) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \lambda^m e^{-2\lambda} \{2\lambda A_m - (3 - 4\eta + 2\lambda) B_m\} \qquad \ldots(31)$$

$$(3 - 4\eta)\,\varphi_1(\lambda) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\,\lambda^m A_m\,(2\lambda - (3 - 4\eta))\,e^{-2\lambda}$$

$$-2\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\,\lambda^{m+1}\,e^{-2\lambda}\,B_m \qquad \ldots(32)$$

which are the conditions under which the free surface of the solid is rigidly clamped.

In order to adjust the boundary conditions on the surface of the cavity we have to expand the stress functions [III] in spherical coordinates valid near the origin. We know[5]

$$e^{-\lambda z}\,J_0(\lambda r) = \sum_{n=0}^{\infty} (-1)^n \frac{(\lambda R)^n}{n!}\,P_n(\mu) \qquad \ldots(33)$$

and with this representation the stress function (III) takes the following from

$$[V]\begin{cases} \varphi_0(\lambda) = p\sum_{n=0}^{\infty} \alpha_n\,R^n\,P_n(\mu) & \ldots(34) \\ \varphi_3(\lambda) = p\sum_{n=0}^{\infty} \beta_m\,R^n\,P_n(\mu) & \ldots(35) \end{cases}$$

where

$$\alpha_n = \int_0^{\infty} \frac{(-1)^n}{n!}\,\lambda^n\,\varphi_1(\lambda)\,d\lambda \qquad \ldots(36)$$

$$\beta_n = \int_0^{\infty} \frac{(-1)^n}{n!}\,\lambda^{n+1}\,\psi_2(\lambda)\,d\lambda. \qquad \ldots(37)$$

Substituting the values of $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$ in (36) and (37) and making use of the following integral :

$$\int_0^{\infty} e^{-ax}\,x^b\,dx = \frac{\Gamma\,b+1}{a^{b+1}}\,,\ (a, b > 0) \qquad \ldots(38)$$

we get

$$(3-4\eta)\,\alpha_n = -\sum_{m=0}^{\infty}\Big[\Big\{(3-4\eta)\,\gamma_n^m + 2(m+1)\,\gamma_n^{m+1}\Big\}A_m - 2(m+1)\,\gamma_n^{m+1}B_m\Big], \quad \ldots(39)$$

$$(3-4\eta)\,\beta_n = -\sum_{m=0}^{\infty}\Big[2(m+1)\,\gamma_n^{m+1}A_m + \Big\{(3-4\eta)\,\gamma_n^m - 2(m+1)\,\gamma_n^{m+1}\Big\}B_m\Big], \quad \ldots(40)$$

where

$$\gamma_n^m = \frac{(-1)^{m+n}}{m!\,n!}\int_0^{\infty}\lambda^{m+n}e^{-2\lambda}\,d\lambda = \frac{(-1)^{m+n}}{m!\,n!}\,\frac{\Gamma(m+n+1)}{2^{m+n+1}}$$

$$\gamma_0^0 = \log e^2. \quad \ldots(41)$$

The stress functions [II], [V] give the following normal and tangential stress on the surface of a spherical cavity

$$\left(\frac{\sigma_R}{p}\right)_{R=a} = \sum_{n=0}^{\infty}\Big[(n+1)(n+2)\,\frac{A_n}{a^{n+3}} + \{n(n+3)-2\eta\}\,\frac{n}{2n-1}\,\frac{B_{n-1}}{a^{n+1}} + \frac{(n+1)(n+2)}{2n+3}\{n+3+2(1-2\eta)\}\,\frac{B_{n+1}}{a^{n+3}} + n(n-1)\,a^{n-1}\alpha_n + \{n-2-2(1-2\eta)\}\,\frac{n(n-1)}{2n-1}\,a^{n-2}\beta_{n-1} + \frac{n+1}{2n+3}\{(n+1)(n-2)-2\eta\}\,a^n\beta_{n+1}\Big]P_n(\mu)$$

$$= -\frac{2}{3}\,[P_0(\mu) - P_2(\mu)] \quad \ldots(42)$$

$$\left(\frac{\tau_{R\psi}}{p\sin\psi}\right)_{R=a} = \sum_{n=1}^{\infty}\Big[\frac{n+2}{a^{n+2}}An + \{n^2-2(1-\eta)\}\,\frac{B_{n-1}}{(2n-1)\,a^{n+1}}$$

(*equation continued on p. 734*)

$$+ \frac{n+2}{2n+3}\{n+3+2(1-2\eta)\}\frac{B_{n+1}}{a^{n+3}} - (n-1)\alpha_n a^{n-2}$$

$$+ \frac{(n+1)(4-4\eta-n)}{2n-1} a^{n-2}\beta_{n-1}$$

$$+ \{2(1-\eta)-(n+1)^2\}\frac{a^n}{2n+3}\beta_{n+1}\Big] \qquad P_n'(\mu) = -\frac{1}{3} P_2'(\mu). \qquad \ldots(43)$$

Equating separately the coefficients of each Legendre function or its derivative to zero in eqns. (42) and (43) we can obtain an infinite set of simultaneous linear algebraic equations. Equation (42) for $n = 0$ gives

$$\frac{2A_0}{a^3} + \frac{2}{3}(5-4\eta)\frac{B_1}{a^3} - \frac{2}{3}(1+\eta)\beta_1 = -\frac{2}{3}; \qquad \ldots(44)$$

for $n = 1$,

$$\frac{6A_1}{a^4} + 2(2-\eta)\frac{B_0}{a^2} + \frac{12}{5}(3-2\eta)\frac{B_2}{a^4} - \frac{4}{5}(1+\eta)a\beta_2 = 0; \qquad \ldots(45)$$

for $n = 2$,

$$\frac{12A_2}{a^5} + \frac{4}{3}(5-\eta)\frac{B_1}{a^3} + \frac{12}{7}(7-4\eta)\frac{B_3}{a^5} + 2\alpha_2$$

$$+ \frac{4}{3}(2\eta-1)\beta_1 - \frac{6}{7}\eta a^2\beta_3 = \frac{2}{3}; \qquad \ldots(46)$$

for $n \geqslant 3$,

$$(n+1)(n+2)\frac{A_n}{a^{n+3}} + \frac{n}{2n-1}\{n(n+3)-2\eta\}\frac{B_{n-1}}{a^{n+1}}$$

$$+ \frac{(n+2)(n+1)}{2n+3}(n+5-4\eta)\frac{B_{n+1}}{a^{n+3}} + n(n-1)a^{n-2}\alpha_n$$

$$+ \frac{n(n-1)}{2n-1}(n-4+4\eta)$$

$$\times a^{n-2}\beta_{n-1} + \frac{n+1}{2n+3}\{(n+1)(n-2)-2\eta\}a^n\beta_{n+1} = 0. \qquad \ldots(47)$$

From eqn. (43) for $n = 1$

$$\frac{3A_1}{a^4} - (1 - 2\eta)\frac{B_0}{a^2} + \frac{6}{5}(3 - 2\eta)\frac{B_2}{a^4} - \frac{2}{5}(1 + \eta)a\beta_2 = 0; \quad \ldots(48)$$

for $n = 2$,

$$\frac{4A_2}{a^5} + \frac{2}{3}(1 + \eta)\frac{B_1}{a^3} + \frac{4}{7}(7 - 4\eta)\frac{B_3}{a^5} - \alpha_2 + \frac{2}{3}(1 - 2\eta)\beta_1 - \frac{7 + 2\eta}{7}a^2\beta_3 = -\frac{1}{3}; \quad \ldots(49)$$

for $n \geqslant 3$

$$(n + 2)\frac{A_n}{a^{n+3}} + \frac{(n^2 - 2 + 2\eta)}{2n - 1}\frac{B_{n-1}}{a^{n+1}} + \frac{n + 2}{2n + 3}(n + 5 - 4\eta)\frac{B_{n+1}}{a^{n+3}} - (n - 1)a^{n-2}\alpha_n + \frac{(n - 1)(4 - 4\eta - n)}{2n - 1}a^{n-2}\beta_{n-1} + \frac{2 - 2\eta - (n + 1)^2}{n + 3}a^n\beta_{n+1} = 0. \quad \ldots(50)$$

From (45) and (48) it can be found that $B_0 = 0$. Substituting the values of α_n and β_n into equations (44) – (50) and solving we can obtain A_n and B_n, $\varphi_1(\lambda)$ and $\varphi_2(\lambda)$.

4. Numerical Results

The numerical calculations are obtained for six values of the radius of the cavity i.e. for $a = 0.2, 0.4, 0.6, 0.7, 0.8$ and 0.9 when $\eta = 1/4$. The results for the coefficients A_n, B_n, α_n, β_n are calculated and it is observed that the values of these coefficients decrease very fast for smaller values of the radius.

It is very easy to find the stresses and displacement at any point in the elastic body. By using the suitable recurrence relations of Legendre Polynomials and the relations [I], [II] and [V] the stress at the surface of a cavity is given by

$$\left(\frac{\sigma_\psi}{p}\right)_{R=a} = \frac{1}{3}(P_0 + 2P_2) + \sum_{n=0}^{\infty}\frac{A_n}{a^{n+3}}\left[P'_{n+1} - (n + 1)(n + 2)P_n\right]$$

$$- \sum_{n=0}^{\infty}\frac{B_n}{(2n + 1)a^{n+2}}\left[(n + 1)\{(n + 1) + 3 - 2\eta\}\right.$$

(*equation continued on p.* 736)

$$\times P_{n+1} + n(n+1)$$

$$\times (n + 4 - 4\eta) P_{n-1} - (2n+1) P'_n \Big] + \sum_{n=0}^{\infty} \alpha_n a^{n-2}$$

$$\Big[P'_{n+1} - (n^2 + n + 1) P_n \Big]$$

$$- \sum_{n=0}^{\infty} \frac{\beta_n a^{n-1}}{2n+1} \Big[n(n^2 + 3n + 3 - 2\eta) P_{n-1} + n(n+1)$$

$$(n - 3 + 4\eta) P_{n+1} - (2n+1) P'_n \Big]. \qquad \ldots(51)$$

Figure 2 graphically represents the result for the stress distribution of σ_ψ at the surface of cavity.

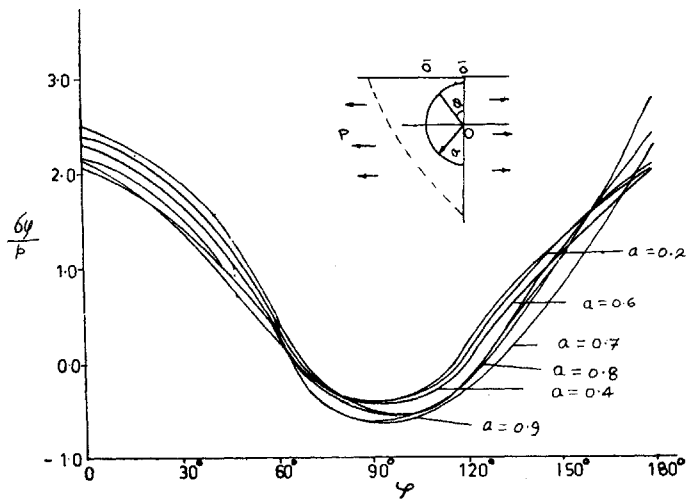

FIG. 2. Stress distribution of σ_φ at the surface of the cavity.

The stress σ_ψ is tensile at $\psi = 0$ and π and becomes compressive near $\psi = 60°$ and again returns tensile near $\psi = 120°$. The maximum compressible stress occurs near $\psi = 90°$. The value of σ_ψ near $\psi = 180°$ is affected by the radius of cavity while it is more affected near $\psi = 0°$. The maximum tensile stress occurs at $\psi = 180°$

The stress σ_θ at the surface of cavity is obtained with the help of equation (4) along with the relation [I], [II], [V] and the suitable recurrence relations of Legendre polynomials we get,

$$\left(\frac{\sigma_\theta}{p}\right)_{R=a} = P_0 - \sum_{n=0}^{\infty} A_n \frac{P'_{n+1}}{a^{n+3}} - \sum_{n=0}^{\infty} \frac{B_n}{a^{n+2}}\left[(1-2\eta)(n+1)P_{n+1} + P'_n\right]$$

$$- \sum_{n=0}^{\infty} \alpha_n a^{n-2} P'_{n-1} + \sum_{n=0}^{\infty}\left[(1-2\eta) n P_{n-1} - P'_n\right] a^{n-1} \beta_n \quad \ldots (52)$$

and its variation is shown in Fig. 3.

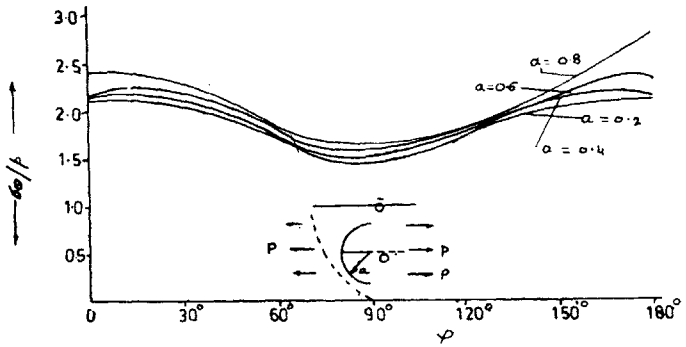

FIG. 3. Stress distribution of $\sigma\theta$ at the surface of the cavity.

The stress σ_θ is tensible all over the surface of a cavity and the minimum tensile stress occurs near $\psi = 90°$. For the stresses σ_ψ, σ_θ along the axis of symmetry we have

$$\left(\frac{\sigma_\psi}{p}\right)_{r=0} = \left(\frac{\sigma_\theta}{p}\right)_{r=0} = 1 - \left[\sum_{n=0}^{\infty} A_n \frac{P'_{n+1}}{R^{n+3}} + \sum_{n=0}^{\infty} \frac{B_n}{R}\right.$$

$$\times \left\{(1-2\eta)(n+1)P_{n+1} + P'_n\right\}$$

$$+ \int_0^{\infty} \varphi_1(\lambda) e^{-\lambda z} \lambda^2 \frac{J_1}{\lambda r} d\lambda + \int_0^{\infty} \lambda^2 \varphi_2(\lambda) e^{-\lambda z}$$

$$\left.\left\{2\eta J_0 - \frac{z J_1}{r}\right\} d\lambda \right]_{r=0}. \quad \ldots (53)$$

We represent the above equation separately at $\psi=0°$ and $\psi = 180°$ by using the following relations

$$P_n(1) = 1,\ P_n'(1) = \frac{n(n+1)}{2},\ \lim \frac{J_1(\lambda r)}{\lambda r} = \frac{1}{2},\ J_0(0) = 1.$$

For the case $\psi = 0°$ we have

$$\left(\frac{\sigma_\theta}{p}\right)_{\psi=0} = 1 - \sum_{n=0}^{\infty} \frac{A_n}{z^{n+3}} \frac{(n+1)(n+2)}{2} - \sum_{n=0}^{\infty} \frac{B_n}{z^{n+2}} (n+1)\left(1-2\eta+\frac{n}{2}\right)$$

$$- \frac{1}{2}\int_0^\infty \lambda^2 e^{-\lambda z} \varphi_1(\lambda)\, d\lambda + \frac{1}{2}\int_0^\infty \lambda^2 \varphi_2(\lambda) e^{-\lambda z}(4\eta-\lambda z)\, d\lambda. \qquad ...(54)$$

Substituting equations (31) and (32) in (54) we have

$$\left(\frac{\sigma_\theta}{p}\right)_{\psi=0} = 1 - \frac{1}{2}\sum_{n=0}^{\infty} \frac{A_n (n+1)(n+2)}{z^{n+3}} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{B_n (n+1)(n+2-4\eta)}{z^{n+2}}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{(3-4\eta)(z+2)^{n+3}} \left[\frac{3-4\eta}{2} + \frac{(n+3)(8\eta-3)}{z+2}\right.$$

$$\left. - \frac{(n+3)(n+4)z}{(z+2)^2}\right] A_n$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)(n+2)}{(3-4\eta)(z+2)^{n+3}} \left[-2\eta(3-4\eta) + (n+3)\right.$$

$$\frac{(1+(3-4\eta)z \quad 4\eta)}{z+2}$$

$$\left. + \frac{(n+3)(n+4)(3-4\eta)z}{(z+2)^2}\right] B_n. \qquad ...(55)$$

For the case $\psi = \pi$, using the relations

$$P_n'(-1) = (-1)^{n+1} \frac{n(n+2)}{2},\ P_n(-1) = (-1)^n \qquad ...(56)$$

we have

$$\left(\frac{\sigma_\theta}{p}\right)_{\psi=\pi} = 1 + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{A_n\,(n+1)\,(n+2)}{|\,z\,|^{n+3}}$$

$$+ \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n}\frac{B_n\,(n+1)\,(n+2-4\eta)}{|\,z\,|^{n+2}}$$

$$+ \sum_{n=0}^{\infty}(-1)^{n}\frac{A_n\,(n+1)\,(n+2)}{(3-4\eta)\,(2-|\,z\,|)^{n+3}}$$

$$\times\left[\frac{3-4\eta}{2} + \frac{(n+3)\,(8\eta-3)}{(2-|\,z\,|)} + \frac{(n+3)\,(n+4)\,|\,z\,|}{(2-|\,z\,|)^2}\right]$$

$$+ \sum_{n=0}^{\infty}(-1)^{n}\frac{(n+1)\,(n+2)}{(3-4\eta)\,(2-|\,z\,|)^{n+3}}\frac{B_n}{}\left[-2\eta\,(3-4\eta)\right.$$

$$\left. + (n+3)\,\frac{\{1-4\eta+(3-4\eta)\,z\}}{(2-|\,z\,|)} - \frac{(n+3)(n+4)(3-4\eta)\,|\,z\,|}{(2-|\,z\,|)^2}\right]. \qquad \ldots(57)$$

The results of $\frac{\sigma_\theta}{p}$ and $\frac{\sigma_\psi}{p}$ at $\psi = 0$ and at $\psi = \pi$ are given in Figs. 4 and 5.

These figures clearly show that the stress at the pole A is affected apparently due to clamping of the plane boundary, while the one at the pole B is considerably affected.

The stress σ_r at the plane boundary is obtained with the help of eqn. (3) and stress functions [I], [II] and [III] with a substitution of (26) in [II] as

$$\left(\frac{\sigma_r}{p}\right)_{z=-1} = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{n\,!}A_n\left[\frac{1}{r}\int_0^{\infty}J_1\,e^{-\lambda}\,\lambda^{n+1}\,d\lambda - \int_0^{\infty}J_0\,e^{-\lambda}\,\lambda^{n+2}\,d\lambda\right]$$

$$+ \sum_{n=0}^{\infty}\frac{(-1)^n}{n\,!}B_n\left[-\frac{1}{r}\int_0^{\infty}J_1\,e^{-\lambda}\,\lambda^{n+1}\,d\lambda + \int_0^{\infty}J_0\,e^{-\lambda}\,\lambda^{n+2}\,d\lambda\right.$$

(equation continued on p. 740)

$$- 2\eta \int_0^\infty J_0\, e^{-\lambda}\, \lambda^{n+1}\, d\lambda \Big] + \int_0^\infty \varphi_1(\lambda)\, \lambda^2\, e^{\lambda} \left(\frac{J_1}{\lambda r} - J_0 \right) d\lambda$$

$$+ \int_0^\infty \varphi_2(\lambda)\, \lambda^2\, e^{\lambda} \left[(2\eta + \lambda)\, J_0 - \frac{J_1}{r} \right] d\lambda. \qquad \ldots(58)$$

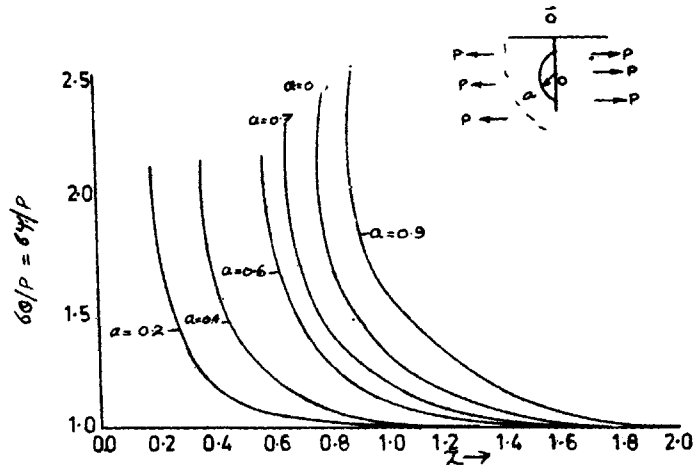

FIG. 4. Stress distribution of $\theta\sigma$ and $\sigma\varphi$ along Z axis.

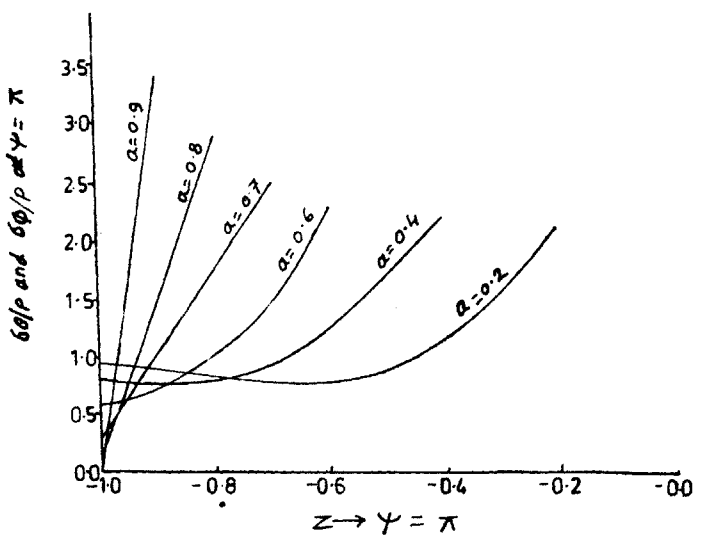

FIG. 5. Stress distribution of $\theta\sigma$ and $\sigma\varphi$ along Z axis.

Substituting the values of $\varphi_1(\lambda)$, $\varphi_2(\lambda)$ from eqns. (31) and (32) into the above equation and using the well known relation.

$$\int_0^\infty J_\nu(at)\, e^{-pt}\, t^\mu\, dt = \frac{\Gamma(\mu - \nu + 1)}{(a^2 + p^2)^{\frac{\mu}{2} + \frac{1}{2}}} P_\mu^\nu\left[\frac{p}{\sqrt{a^2 + p^2}}\right],$$

$$\left(\frac{\sigma_r}{p}\right)_{z=-1} = 1 + 2\sum_{n=0}^{\infty} (-1)^n \frac{A_n (n+1)(n+2)}{(3-4\eta) R_0^{n+3}}$$

$$\left\{\frac{1}{r}\left(P'_{n+2}(\mu_0) - \frac{n+3}{R_0} P'_{n+3}\mu_0\right)\right.$$

$$\left. - \frac{1-2\eta}{R_0} P_{n+3}(\mu_0) + \frac{(n+3)(n+4)}{R_0^2} P_{n+4}(\mu_0)\right\}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n (n+1) B_n}{(3-4\eta) R_0^{n+2}} \left\{\frac{1}{r}\left[-(3-4\eta) P'_{n+1}(\mu_0)\right.\right.$$

$$+ \frac{(n+2)(1-4\eta)+2}{R_0} \times P'_{n+2}(\mu_0)$$

$$\left. + \frac{2(n+2)(n+3)}{R_0^2} P'_{n+3}(\mu_0)\right] + \left[-2\eta(3-4\eta) P_{n+1}(\mu_0)\right.$$

$$+ \frac{(n+2)(1-2\eta)(3-4\eta)}{R_0} P_{n+2}(\mu_0) - \frac{(n+2)(n+3)}{R_0^2} P_{n+3}(\mu_0)$$

$$\left.\left. - \frac{2(n+2)(n+3)(n+4)}{R_0^3} P_{n+4}(\mu_0)\right]\right\} \qquad \ldots(59)$$

where $R_0 = \sqrt{1 + r^2}$, $\mu_0 = R_0^{-1}$ and P_n^m is the Legendre associated function of degree n and order m. Also for the stress σ_θ at the plane boundary we have

$$\left(\frac{\sigma_\theta}{p}\right)_{z=-1} = 1 - \sum_{n=0}^{\infty} (-1)^n \frac{2(n+1)(n+2)}{(3-4\eta) R_0^{n+3}} A_n \left\{\frac{1}{r}\left(P'_{n+2}(\mu_0)\right.\right.$$

$$+ \frac{n+3}{R_0} P'_{n+3}(\mu_0)\Big) + \frac{2\eta(n+3)}{R_0} P_{n+3}(\mu_0)\Big\}$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n 2(n+1) B_n}{(3-4\eta) R_0^{n+2}} \Big[\frac{n+2}{r R_0}\Big(P'_{n+2}(\mu_0)$$

$$\times 4(1-\eta) + \frac{n+3}{R_0} P'_{n+3}(\mu_0)\Big) + (3-4\eta)\eta\Big(-P_{n+1}(\mu_0)$$

$$+ \frac{n+2}{R_0} P_{n+2}(\mu_0)\Big) + \frac{(n+2)(n+3)}{R_0^2} P_{n+3}(\mu_0)\Big]. \quad \ldots(60)$$

The result of σ_r and σ_θ at the plane boundary are shown in Figs. 6 and 7 respectively

the maximum tensile stress in the semi-infinite body with clamped edge occurs on the surface of the cavity at $\psi = \pi$. Figure 8 illustrate the relation between the maximum tensile stress and the radius of the cavity.

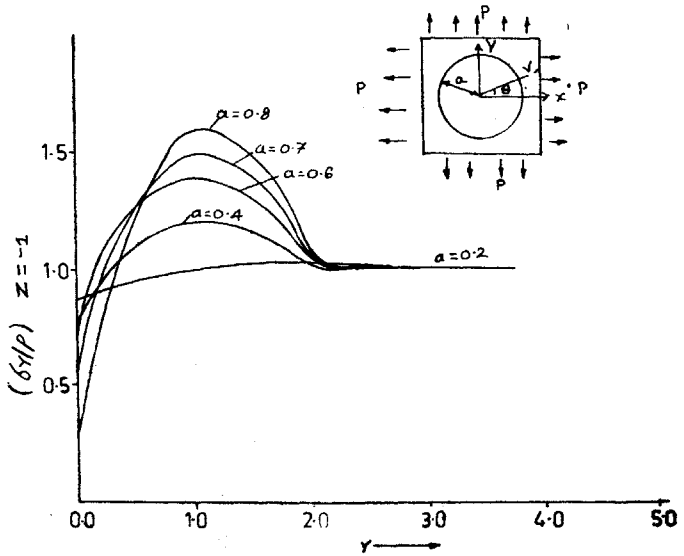

FIG. 6. Stress distribution of σr on the plane boundary.

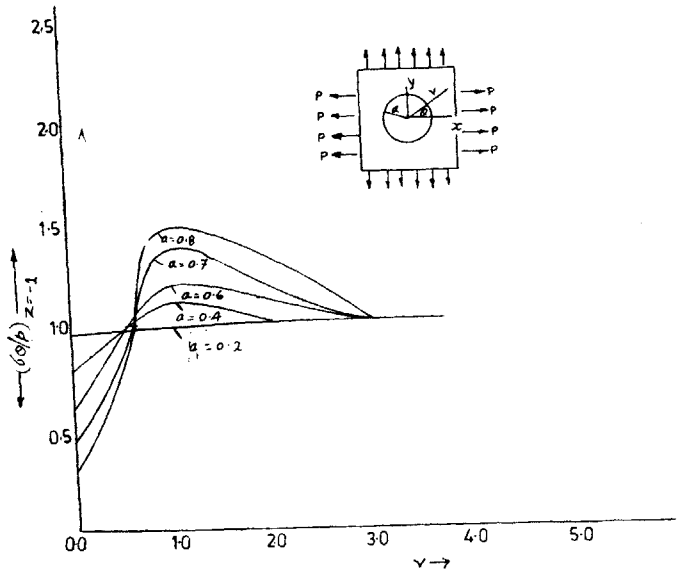

FIG. 7. Stress distribution of σ_θ on the plane boundary.

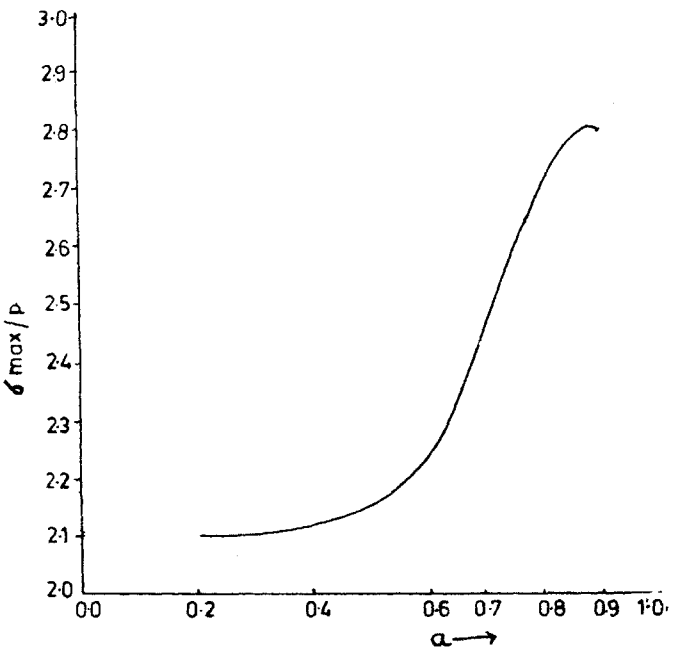

FIG. 8. Radius-of-the-cavity.

Conclusions

The present paper is the investigation of axisymmetric problem of semi-infinite solid with clamped surface suiting the stability of Engineering problems in the fields. The problem has been solved by using cylindrical and spherical harmonics. We observe here that the use of cylindrical harmonics, in an infinite integrals, gives us good convergence for the coefficients of infinite series. Numerical results for the stress distribution on the plane boundary and on the surface of cavity are shown graphically. It is observed that with an increase of radius of the cavity maximum tensile stress occurs at $\psi = \pi$ on the surface of cavity and as $R \to \infty$ stress vanishes on the plane boundary when $\theta \to 0$. These characters are analogous to the results of two dimensional problem containing circular hole[6].

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