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CONNECTEDNESS AND ACYCLICITY OF THE TRANSPOSED SQUARE OF A GRAPH

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Plonka[2] introduced the notion of the transposed square of a given graph, possibly with loops. In this paper, we characterize connected (acyclic) graphs, without loops, whose transposed squares are connected (acyclic).

For all terminology and notation in graph theory we refer the reader to Harary[1].

Given a graph $G = (V, E)$, its 'transposed square', or t-square, $G\llcorner^2\urcorner$ is obtained on the vertex set $V \times V$ by defining adjacency between any two ordered pairs (a, b) and (c, d) of vertices of G whenever ad and bc are both edges of G (cf. Plonka[2]). For example, the t-square of K_2 is shown in Fig. 1.

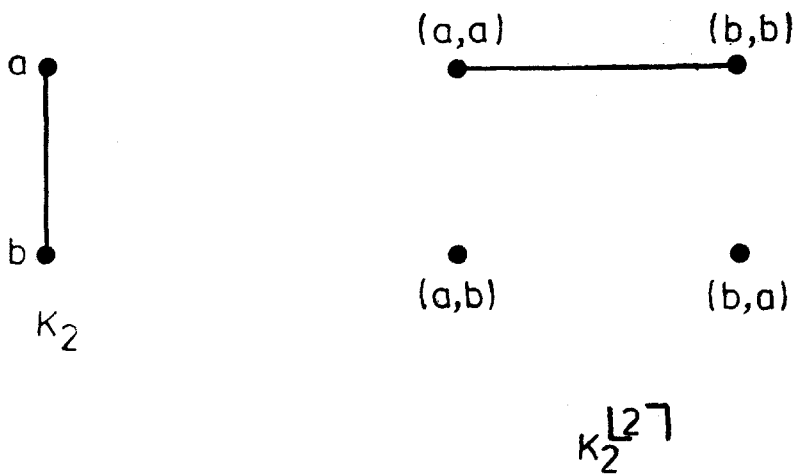

FIG. 1.

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While Plonka[2] considers 'pseudographs' (i.e. graphs in which loops are allowed at some or all vertices), we shall restrict our investigation only to graphs without loops. We caution the reader here that this restriction makes lot of difference.

In any graph $G = (V, E)$, the fact that two vertices u and v are adjacent shall henceforth be indicated symbolically as $a \leftrightarrow b$. Recall that a sequence $(a_0, a_1, a_2, \ldots, a_k)$, $k \geqslant 0$, of vertices of G is called a *chain* if $a_0 \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow \ldots \leftrightarrow a_k$ and the number k is called the *length* of the chain. We may say that a chain $(a_0, a_1, a_2, \ldots, a_k)$ *connects* a and b if $a = a_0$ and $b = a_k$. G is said to be connected if any two of its vertices are connected by a chain. The maximal connected subgraph of G forms a *component* of G. G is said to be *disconnected* if it has at least two components. The following two observations, theorem and its corollary due to Plonka[2] are needed to establish our results here.

01. $(a, b) \leftrightarrow (c, d)$ in $G\llcorner^2\urcorner \leftrightarrow (b, a) \leftrightarrow (d, c)$ in $G\llcorner^2\urcorner$ (this property is called the 'symmetry property' of adjacency in $G\llcorner^2\urcorner$).

02. For every graph $G = (V, E)$, the subgraph of $G\llcorner^2\urcorner$ induced by the set $\{(a, a) | a \in V)\}$, called the 'diagonal' is isomorphic to G and is denoted as $G^{(2)}$.

Theorem 1 (Plonka[2])—Given a graph G, there exists a chain H connecting vartices (a, b) and (c,d) in the t-square of G if and only if one of the following conditions (a),(b) and (c) holds :

(a) : $a = c$ and $b = d$

(b) : There exists a chain H_1 of an odd length connecting a and d and a chain H_2 of an odd length connecting b and c in G

(c) : There exists a chain H_3 of a positive even length connecting a and c and a chain H_4 of a positive even length connecting b and d in G.

Corollary 1.1 (Plonka[2])—The t-square of the cycle C_n of length n is connected if and only if n is odd.

In this note, we generalize the statement of Corollary 1.1 as follows :

Theorem 2—The t-square of a connected graph G is connected if and only if G contains a cycle of odd length (i.e. iff G is nonbipartite).

PROOF : Let $G = (V, E)$ be a bipartite graph with $\{V_1, V_2\}$ as its bipartition. Let $a \in V_1$ and $b \in V_2$. Since G is connected a and b must be connected by at least one chain in G, but all such ab-chains in G must be of odd lengths as G is bipartite. Since a is connected to itself by a chain of length 0 (which is even but not positive

even) it follows, therefore, that conditions (b) and (c) of Theorem 1 do not hold for the vertices (a, a) and (a, b) of $G\llcorner^2\urcorner$. Also, condition (a) of Theorem 1 does not hold for this pair of vertices of the t-square of G. Thus, by Theorem 1 it follows that (a, a) and (a, b) are not connected by any chain in $G\llcorner^2\urcorner$ and hence they must be in different components of $G\llcorner^2\urcorner$. This shows that the t-square of G is disconnected when G is bipartite.

For the converse, assume that G has an odd cycle, say $C_n = (a_1, a_2,\ldots,a_n, a_1)$, $a_i \in \underline{n} = \{1, 2,\ldots,n\}$, with $a_i a_{i+1} \in E(G)$ for each $i \in \underline{n}$ where indices are reduced modulo n. By property 02, G is isomorphic to the subgraph $G^{(2)}$ of $G\llcorner^2\urcorner$ induced by the diagonal $\{(a, a)/a \in V\}$ and hence contains the odd cycle $C_n^{(2)} = ((a_1, a_1), (a_2, a_2), \ldots, (a_n, a_n), (a_1,a_1))$. Now, consider any vertex (a, b) of $G\llcorner^2\urcorner$. Then we shall show that (a, b) is connected by a chain to some vertex (a_i, a_i) of $C_n^{(2)}$. Two cases must be considered.

Case 1 : At least one of a and b lies on C_n in G

Without loss of generality, let a lie on C_n, i.e. $a = a_i$ for some $i \in \underline{n}$. Also, we may take $i = 1$ for otherwise we may relabel the vertices of C_n so as to have this situation. Since G is connected there is a ba_1-path, say $P = (b = b_0, b_1, b_2,\ldots,b_k = a_1)$ in G.

If k is even then $((a_1 = a, b_0 = b), (b_1, a_2), (a_1, b_2), (b_2, a_2), (a_1, b_4),\ldots, (b_{k-1}, a_2), (a_1, b_0 = a_1))$ is an (a, b) (a_1, a_1)-chain in $G\llcorner^2\urcorner$. If k is odd then the following is such a chain in the t-square of G :

$$((a, b), (b_1, a_2), (a_1, b_2), (b_2, a_2),\ldots, (b_k = a_1, a_2),$$

$$(a_3, b_{k-1}), (b_{k-2}, a_2), (a_3, b_{k-3}),\ldots, (a_3, b_0 = b),$$

$$(b_1, a_4), (a_2, b_2), \ldots, (b_2, a_4)\,(b_k = a_1, a_4)$$

$$(a_5, b_{k-1}), (b_{k-2}, a_5), (a_5, b_{k-3}), \quad , (a_5, b_0 = a_1) ,\ldots,$$

$$(a_n, b_{k-1}), (b_{k-2}, a_{n-1}), (a_n, b_{k-3}),\ldots, (a_n, b_0 = b),$$

$$(b_1, a_1), (a_n, b_2), (b_3, a_1),\ldots, (b_k = a_1, a_1)).$$

Case 2 : Both a and b are not on C_n in G

Since G is connected a and b are connected by some chains in G to some vertices a_i and a_r on C_n, respectively, with i and r being possibly equal. Without loss of generality, we may assume $i = 1$. Let $P = (a = x_0, x_1, x_2,\ldots, x_k, = a_1)$ and $Q = (b = y_0, y_1, y_2,\ldots,y_t = a_r)$ be some aa_1 – and ba_r-chains in G. Again, without loss to generality, we may assume that $1 \leqslant k \leqslant t$. Hence, by Case 1 it follows

that there must exist an (a_1, b) (a_1, a_1) chain H_1 in the t-square of G. If k is even then $H_2 : = ((a, b), (y_1, x_1), (x_2, y_2), (y_3, x_3), \ldots, (x_k, y_k), (y_{k-1}, a_2), (a_1, y_{k-2}), (y_{k-3}, a_2), \ldots, (a_1, y_0 = b))$ is an (a, b) (a_1, b)-chain in $G\llcorner^2\urcorner$. Hence, the union of H_1 and H_2 contains an (a, b) (a_1, a_1)-chain. On the other hand, if k is odd then $H_3 : = ((a, b), (y_1, x_1), (x_2, y_2), \ldots, (y_k, x_k = a_1), (a_2, y_{k-1}), (y_{k-2}, a_1), (a_2, y_{k-3}), \ldots, (b = y_0, a_1))$ is an (a, b) (b, a_1)- chain in the t-square of G. Invoking the property of symmetry 01, we may find a (b, a_1) (a_1, a_1)-chain H_1' in $G\llcorner^2\urcorner$, and hence the union of H_3 and H_1' must contain an (a, b) (a_1, a_1)-chain in $G\llcorner^2\urcorner$.

Plonka[2] observes that the t-square of an acyclic graph may not be acyclic and cites the example of P_5, the simple path of length 4. Since every graph is isomorphic to an induced subgraph of its t-square (cf. 02) we then see that P_5 is a forbidden subgraph for any graph with acyclic t-square. 02 also implies that any graph whose t-square is acyclic must itself be acyclic. Precisely which acyclic graphs have acyclic t-squares ? This question is answered in the following theorem.

Theorem 3—For any graph G, $G\llcorner^2\urcorner$ is acyclic if and only if each component of G is a path of length $k \leqslant 3$.

PROOF : Clearly, it is enough to prove the result for a connected graph. If G is P_k for $k \leqslant 4$, it is easy to verify that $G\llcorner^2\urcorner$ is acyclic.

For the converse, assume that $G\llcorner^2\urcorner$ is acyclic. Then G cannot contain a vertex of degree $\geqslant 3$ for if there is a vertex a adjacent to three other vertices x, y, z then $((a, x), (a, y), (a, z), (a, x))$ is a cycle in $G\llcorner^2\urcorner$, a contradiction. Since G is connected, by supposition, it then follows that $G = P_k$ for some positive integer k. Hence, as per our observation above, we must have $k \leqslant 4$ and hence the assertion follows.

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