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PERSPECTIVITY AND PROJECTIVITY IN N.T.C. AND W-MODULAR LATTICES

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In this paper we characterize non-trivial congruent lattices with regard to perspectivity and projectivity. Under certain conditions we prove existence of meet separating elements in N.T.C. lattices. Then it is proved that in a finite N.T.C. lattice any two quotients are connected. Finally we prove that in a lattice satisfying certain condition an element is standard if it is neutral.

§ 1. In this paper we characterize non-trivial congruent lattices with regard to 'perspectivity and projectivity'. Existence of certain useful elements has been proved in section 3. Finally in section 4, we study the concept of standard and neutral elements.

We shall be freely using the notations and conventions of Grätzer[1]. In particular, we wish to call attention to the notations for non-trivial-congruence lattices (n.t.c. lattices) and a class $\mathscr{L}$ of lattices in which for each lattice $\frac{a}{b} \approx \frac{c}{d}$ and $\frac{a}{b}$ is a sub-quotient of $\frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d} \Leftrightarrow a = c$ and $b = d$. Let L be a lattice and $L_c^2 = \{< a, b > \mid a, b \in L \text{ and } b \leqslant a\}$ and

$$L^Q = \left\{ \frac{a}{b} \mid b \leqslant a ; a, b \in L \right\}.$$

Apart from these we make the following :

1. Call a lattice an n.t.c. lattice iff $\theta(a, b) \neq w$ (w is a trivial congruence relation) for all $a, b \in L$.

Remark 1 : As done (Grätzer[1]) L_c^2 is a poset under partial order '$\leqslant$' defined as

$$< a, b > \leqslant < c, d > \Leftrightarrow a \leqslant c, b \leqslant d.$$

Moreover L_c^2 is a lattice if we define $\wedge$ and $\vee$ as

(i) $< a, b > \wedge < c, d > = < a \wedge c, b \wedge d >$

(ii) $< a, b > \vee < c, d > = < a \vee c, b \vee d >$.

Remark 2 : Set $L^Q = \left\{ \frac{a}{b} \mid b \leqslant a ; a, b \in L \right\}$ is a poset if partial order in L^Q is defined as

$$\frac{a}{b} \leqslant \frac{c}{d} \Leftrightarrow a \leqslant c \text{ and } b \leqslant d.$$

Moreover, L^Q is a lattice if we define $\wedge$ and $\vee$ as

(i) $\frac{a}{b} \wedge \frac{c}{d} = \frac{a \wedge c}{b \wedge d}$

(ii) $\frac{a}{b} \vee \frac{c}{d} = \frac{a \vee c}{b \vee d}$.

§ 2. In this section we characterise a lattice with respect to projectivity and establish isomorphism between L_c^2 and L^Q.

Theorem 2.1—Let L be a lattice. The map $\phi : L_c^2 \to L^Q$ defined by $\phi < a, b > = \frac{a}{b}$ is an isomorphism.

PROOF : Let $\phi < a, b > = \phi < c, d >$

$$\Leftrightarrow \frac{a}{b} = \frac{c}{d} \Leftrightarrow a = c, b = d$$

$$\Leftrightarrow < a, b > = < c, d >$$

hence ϕ is one-one.

Let $\frac{c}{d} \in L^Q$. Obviously, $< c, d > \in L_c^2$ and $\phi < c, d > = \frac{c}{d}$, showing that ϕ is onto.

$$\phi (< a, b > \wedge < c, d >) = \phi (< a \wedge c, b \wedge d >)$$

$$= \frac{a \wedge c}{b \wedge d} = \frac{a}{b} \wedge \frac{c}{d} = \phi < a, b > \wedge \phi < c, d >.$$

ϕ preserves $\wedge$, similarly one can prove that ϕ preserves $\vee$.

Q.E.D.

Corollary 2.2—Let $\frac{a_1}{a_2}$, $\frac{b_1}{b_2}$, $\frac{c_1}{c_2}$ and $\frac{d_1}{d_2} \in L^Q$. Then

$$\begin{matrix} a_1 & c_1 \\ \overline{a_2} & \overline{c_2} \\ \overline{b_1} & \overline{d_1} \\ \overline{b_2} & \overline{d_2} \end{matrix} \approx w \; \frac{a_1}{b_1} \approx w \; \frac{c_1}{d_1} \Rightarrow \phi < a_1, b_1 > \approx w \; \phi < c_1, d_1 >$$

$\forall\ a_2, b_2, c_2$ and $d_2 \in L$.

PROOF : By virtue of the theorem just proved, identify $\frac{a_1}{a_2} \in L^Q$ with $< a_1, a_2 >$ of L_c^2 etc.

$$\begin{matrix} a_1 & c_1 \\ \overline{a_2} & \overline{c_2} \\ \overline{b_1} & \overline{d_1} \\ \overline{b_2} & \overline{d_2} \end{matrix} \approx w \; \frac{< a_1, a_2 >}{< b_1, b_2 >} \approx w \; \frac{< c_1, c_2 >}{< d_1, d_2 >}$$

$$\Rightarrow \frac{a_1}{c_1} \approx w \; \frac{b_1}{d_1}, \text{ by Lemma 5 in Grätzer[1], (p. 154),}$$

i.e.

$$\phi < a_1, c_1 > \approx w \; \phi < b_1, d_1 > \forall\ a_2, b_2, c_2, d_2 \in L.$$

Q.E.D.

Theorem 2.3—In any lattice L proper quotient $\frac{a}{b}$ and $\frac{c}{d}$ such that $\frac{a}{b} \approx \frac{c}{d}$ and $b \lessdot a \leqslant \alpha \leqslant d' \leqslant c'$ do not exist, where $\frac{c}{d}$ is a subquotients of $\frac{c'}{d'}$ and α is a distributive element of L.

PROOF : $\frac{a}{b} \approx \frac{c}{d} \Leftrightarrow \frac{a}{b} \approx w \; \frac{c'}{d'}$, where $\frac{c}{d}$ is subquotient of $\frac{c'}{d'}$ i.e.

$$d' \leqslant d \lessdot c \leqslant c'. \qquad \ldots(1)$$

If

$$b \lessdot a \leqslant \alpha \leqslant d' \leqslant c'. \qquad \ldots(2)$$

Then by Lemma 7 in Grätzer[1], (p. 142), $a = b$.

Combining (1) and (2) we get

$$b \leqslant a \leqslant \alpha \lessdot d' \leqslant d \lessdot c \leqslant c'$$

hence $b \leqslant a \leqslant \alpha \lessdot d' \leqslant c'$, $\frac{a}{b} \approx w \frac{c'}{d'}$ and $a = b \Rightarrow c' = d'$. By (1) $c = d$. Thus we have $a = b$ and $c = d$, which contradicts the assumption that $\frac{a}{b}$ and $\frac{c}{d}$ are proper quotients.

Q.E.D.

§ 3. In this section we characterize n.t.c. lattices and show existence of an element d corresponding to an arbitrary pair of elements a, b in any n.t.c. lattice such such that $\frac{a \wedge d}{b \wedge d} \approx w \frac{a}{b}$. We start by proving :

Theorem 3.1—An n.t.c. modular lattice $\in$ class $\mathscr{L}$ does not have a proper quotient $\frac{a}{b}$ which admits of a subquotient $\frac{c}{d}$ such that $\frac{a}{b} \approx w \frac{c}{d}$.

PROOF : On contrary let $\frac{c}{d}$ be a subquotient of $\frac{a}{b}$.

$$b \leqslant d \leqslant c \leqslant a. \qquad \ldots(1)$$

Now, since L is an n.t.c. modular lattice $\therefore \theta(c, d) \neq w$ and there exists a proper quotient $\frac{a}{b}$ of L such that $\frac{a}{b} \approx w \frac{c}{d} \Rightarrow$ there exists, a subquotient $\frac{c'}{d'}$ of $\frac{c}{d}$ such that

$$\frac{a}{b} \approx \frac{c'}{d'} \qquad \ldots(A)$$

[by Exercise 3 in Grätzer[1] (p. 135)], hence

$$d \lessdot d' \leqslant c' \lessdot c. \qquad \ldots(2)$$

Combining (1) and (2) we have

$$b \leqslant d \leqslant d' \leqslant c' \leqslant c \leqslant a \qquad \ldots(P)$$

$$\Rightarrow b \leqslant d' \leqslant c' \leqslant a \Rightarrow \frac{c'}{d'} \text{ is a subquotient of } \frac{a}{b} \qquad \ldots(B)$$

also since $L \in \mathscr{L} \therefore$ from (A) and (B)

it follows that $\frac{a}{b} = \frac{c'}{d'} \Leftrightarrow a = c'$ and $b = d'$

from (P), we get $a = c = c'$ and $b = d = d'$

which contradicts the assumption that $\frac{a}{b}$ is a proper subquotient of L.

Q.E.D.

Theorem 3.2—Let L be an n.t.c. lattice. If for any two elements $a, b \in L$

$$a \wedge (e \vee f) = (a \wedge e) \vee f$$

and

$$b \wedge (e \vee f) = (b \wedge e) \vee f \; \forall \; e, f \in L.$$

Then there exists an element $d \in L$ such that $\frac{a \wedge d}{b \wedge d} \approx w \frac{a}{b}$.

PROOF : Since L is an n.t.c. lattice hence $\theta(a, b) \neq w \; \forall \; a, b \in L$. Therefore there exists a proper quotient $\frac{p}{q}$ in L such that

$$\frac{p}{q} \approx w \frac{a}{b} \qquad \ldots(A)$$

$\Rightarrow$ there exists an integer m and elements $e_0, e_1, \ldots, e_{m-1}$ such that

$$p_m(a, e_0, e_1, \ldots, e_{m-1}) = p \qquad \ldots(1)$$

and

$$p_m(b, e_0, e_1, \ldots, e_{m-1}) = q \qquad \ldots(2)$$

where polynomial p_m is defined by

$$p_m(x_0, x_1, x_2, \ldots, x_n) = (((x_0 \wedge x_1) \vee x_2) \wedge x_3) \vee \ldots)$$

now by using Ist condition assumed in the theorem we get

$$((a \wedge e_0) \vee e_1) = a \wedge (e_0 \vee e_1)$$

and

$$(((a \wedge e_0) \vee e_1) \wedge e_2) = (a \wedge (e_0 \vee e_1) \wedge e_2)$$

$$= a \wedge ((e_0 \vee e_1) \wedge e_2)$$

we go on likewise to get

$$(\ldots (((a \wedge e_0) \vee e_1) \wedge e_2) \vee \ldots)$$

$$= a \wedge (\ldots ((e_0 \vee e_1) \wedge e_2) \vee \ldots) = a \wedge d$$

where

$$d = (\ldots ((e_0 \vee e_1) \wedge e_2) \vee e_3) \ldots).$$

Similarly by using the second condition assumed in the theorem we can get

$$b \wedge (\ldots ((e_0 \vee e_1) \wedge e_2) \vee \ldots) = q$$

i.e.

$$b \wedge d = q.$$

Thus the existence of an element $d \in L$ is proved satisfying

$$a \wedge d = p \text{ and } b \wedge d = q$$

$\therefore$ (A) Takes the form

$$\frac{a \wedge d}{b \wedge d} \approx_w \frac{a}{b}.$$

Q.E.D.

Recall a definition : An element d of a lattice is said to be $\wedge$-separating element if $a \wedge d = b \wedge d \Rightarrow a = b$.

Corollary 3.3—Element 'd' (of Theorem 3.2) is $\wedge$ -separating.

PROOF : We proved in Theorem 3.2 that $\frac{a \wedge d}{b \wedge d} \approx_w \frac{a}{b}$.

$$a \wedge d = b \wedge d \Rightarrow a = b.$$

Q.E.D.

For a distributive element d of a lattice L, $L \wedge d = \{a \wedge d \mid a \in L\}$ is a poset if partial ordering is defined by $a \wedge d \leqslant b \wedge d$ iff $a \leqslant b$ and is a lattice if $\wedge$ and $\vee$ are defined by $(a \wedge d) \wedge (b \wedge d) = (a \wedge b) \wedge d$ and $(a \wedge d) \vee (b \wedge d) = (a \vee b) \wedge d$.

Corollary 3.4—If 'd' (of Theorem 3.2) is distributive element of a lattice L then the map $\phi : \rightarrow L \wedge d$, defined by $\phi(a) = a \wedge d$ is an isomorphism.

PROOF : ϕ is one-one follows from Corollary 3.3.

Obviously ϕ is onto.

$\phi(a \wedge b) = (a \wedge b) \wedge d = (a \wedge d) \wedge (b \wedge d) = \phi(a) \wedge \phi(b)$, hence ϕ is $\wedge$-preserving.

$$\phi(a \vee b) = (a \vee b) \wedge d = (a \wedge d) \vee (b \wedge d) = \phi(a) \vee \phi(b)$$

($\because$ d is distributive).

Thus, ϕ is an isomorphism of L onto $L \wedge d$.

Q.E.D.

Corollary 3.5—For any two elements $a, b \in$ an n.t.c. lattice L, satisfying

$$\text{(i) } a \wedge (e \vee f) = (a \wedge e) \vee f \text{ and (ii) } b \wedge (e \vee f) = (b \wedge e) \vee f$$

$$\forall\ e, f \in L.$$

There exist $c, c' \in L$ such that $c \wedge d = a$ and $c' \wedge d = b$ and $\dfrac{a \wedge d}{b \wedge d} \approx_w \dfrac{a}{b} \approx_w \dfrac{a \vee c}{b \vee c'}$, where d is a distributive element of Theorem 3.2.

PROOF : Existence of elements c and c' is clear in view of isomorphism ϕ in Corollary 3.4.

Theorem 3.2 guarantees existence of $d \in L$ such that

$$\frac{a \wedge d}{b \wedge d} \approx_w \frac{a}{b} \qquad \text{...(A)}$$

$$\phi(a \vee c) = (a \vee c) \wedge d = (a \wedge d) \vee (c \wedge d) \ (\because d \text{ is distributive})$$

$$= (a \wedge d) \vee a = a \ (\because c \wedge d = a)$$

and

$$\phi(b \vee c') = (b \vee c') \wedge d = (b \wedge d) \vee (c' \wedge d) \ (\because d \text{ is distributive})$$

$$= (b \wedge d) \vee b = b \ (\because c' \wedge d = b).$$

Thus

$$\text{(A)} \Rightarrow \frac{\phi(a)}{\phi(b)} \approx_w \frac{a}{b} = \frac{\phi(a \vee c)}{\phi(b \vee c')}.$$

Now by Exercise 20, (p. 136, [Grätzer[1]]) there exists a quotient $\dfrac{a}{b}$ of L with

$$\frac{a}{b} \approx_w \frac{a \vee c}{b \vee c'}. \qquad \text{...(B)}$$

Combining (A) and (B) we get

$$\frac{a \wedge d}{b \wedge d} \approx_w \frac{a}{b} \approx_w \frac{a \vee b}{b \vee c'}.$$

Q.E.D.

We shall use the following definition in the next theorem.

Definition—We call two quotients $\frac{a}{b}$ and $\frac{c}{d}$ connected iff there exists a finite number of quotients $\frac{p_i}{q_i}$, $0 \lessdot i \leqslant n$ such that

$$\frac{a}{b} = \frac{p_0}{q_0} \approx_w \frac{p_1}{q_1} \approx_w \ldots \approx_w \frac{p_n}{q_n} = \frac{c}{d}.$$

Theorem 3.6—In a finite n.t.c lattice any two quotients are connected.

PROOF : Let $a, b, p, q \in L$ such that $b < a$ and $q < p$

$\theta(a, b) \neq w \Rightarrow$ there exists a proper quotient

$$\frac{c}{d} \text{ such that } \frac{c}{d} \approx_w \frac{a}{b}. \qquad \ldots(1)$$

Again $\because \theta(c, d) \neq w$, hence there exists a proper quotient

$$\frac{p}{q} \text{ such that } \frac{p}{q} \approx_w \frac{c}{d}. \qquad \ldots(2)$$

(1) and (2) $\Rightarrow \frac{p}{q} \approx_w \frac{c}{d} \approx_w \frac{a}{b}$.

On carrying this argument successively we get

$$(*) \quad \ldots \approx_w \frac{r'}{s'} \approx_w \frac{p}{q} \approx_w \frac{c}{d} \approx_w \frac{a}{b}.$$

Now, since L is finite, only a finite number of different quotients can be formed. Hence chains of type (*) must terminate. This proves that quotients $\frac{a}{b}$ and $\frac{p}{q}$ are connected.

Q.E.D.

§ 4. In this section we generalise the results of Grätzer and Schmidt [See Theorem 6 in Grätzer[1] (p. 142), and Lemma 7 in Grätzer[1] (p. 142)]. Results have been proved—replacing 'distributive' elements by 'standard' elements. We begin with.

Theorem 4.1—Let L be a lattice and $a \in L$. The element 'a' is dually standard if for some $p \in L$, $u \leqslant z \lessdot a \wedge p \leqslant y \leqslant x$ and $\frac{x}{y} \approx_w \frac{z}{u} \Rightarrow x = y$.

PROOF : Assume $u \lessdot z \leqslant a \wedge p \lessdot y \leqslant x$ and $\frac{x}{y} \approx_w \frac{z}{u} \Rightarrow x = y$.

If 'a' is not dually standard then $b \vee (a \wedge c) \neq (b \vee a) \wedge (b \vee c)$; $b, c \in L$. Observe that $a \equiv (a \wedge b \wedge c)\,(\theta_a)$, where (θ_a) is a congruence relation, defined in Theorem 2 (p. 139, Grätzer[1]) : $x \equiv y\,(\theta_a)$ iff $x \vee a = y \vee a$.

$$b \vee (a \wedge c) \equiv (b \vee ((a \wedge b \wedge c) \wedge c))\,(\theta_a) = (b \vee (a \wedge b \wedge c))\,(\theta_a) = b\theta_a).$$

Also $(b \vee a) \wedge (b \vee c) \equiv ((b \vee (a \wedge b \wedge c)) \wedge (b \vee c))\,(\theta_a) = (b \wedge (b \vee c))\,(\theta_a) = b\,(\theta_a)$.

Hence $b \vee (a \wedge c) \equiv ((b \vee a) \wedge (b \vee c)\,(\theta_a)$.

By Corollary 5 (p. 132, Grätzer[1]), there exist $x, y, u \in L$, satisfying with $z = a \wedge (b \vee c)$,

$$u < z \leqslant a \wedge (b \vee c) \leqslant (b \vee a) \wedge (b \vee c) \leqslant y < x \leqslant b \vee (a \wedge c),$$

$\frac{x}{y} \approx w \frac{z}{u}$, which is a contradiction. Hence 'a' is dually standard and element of L.

Theorem 4.2—Let 'a' be a standard element of a lattice L. Then for all $p \in L$

$$u \leqslant z \leqslant a \wedge p \leqslant y \leqslant x \text{ and } \frac{x}{y} \approx w \frac{z}{u} \Rightarrow x = y.$$

PROOF : $u \leqslant z \Rightarrow u \vee z = z \leqslant a \wedge p \leqslant a \Rightarrow u \equiv z(\theta_a)$, where (θ_a) is a congruence relation as defined in Theorem 3 (p. 139, Grätzer[1]).

Now, since $u \equiv z\,(\theta_a)$ and $\frac{x}{y} \approx w \frac{z}{u} \Rightarrow x \equiv y\,(\theta_a) \Rightarrow (x \wedge y) \vee q = x \vee y$ for some $q \leqslant a$. We have $y \vee q = x$. Since $a \wedge p \leqslant y$ for all $p \in L$, hence in particular $q = a \wedge q \leqslant y \Rightarrow y \vee q = x$ is equivalent to $y = x$.

Theorem 4.3— Let L be lattice s.t. for $p \in L$, $u \leqslant z \leqslant a \wedge p \leqslant y \leqslant x$ and $\frac{x}{y} \approx w \frac{z}{u}$ $\Rightarrow x = y$; $x, y, z, u, a \in L$. Then an element 'a' of L is standard iff it is neutral.

PROOF : 'a' $\in L$ is neutral $\Rightarrow$ 'a' is standard.

Conversely, let 'a' be a standard element of L. $\Rightarrow$ 'a' is distributive. By Theorem 4.1, 'a' is dually standard $\Rightarrow$ 'a' is dually distributive. Now since 'a' is distributive and dually distributive, hence it is enough to prove that

$$a \wedge x = a \wedge y \text{ and } a \vee x = a \vee y \Rightarrow x = y.$$

Assume the contrary that $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ but $x \neq y$ for some $x, y \in L$. The rest of the proof is similar to the proof of Theorem 6 in Grätzer[1] (p. 142).

REFERENCE

1. G. Grätzer, *General Lattice Theory*, Birkhauser-Verlag, Basel, 1978.