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ON CONVEX SUBLATTICES

V. N. DIXIT AND D. K. PALIWAL
Department of Mathematics, University of Delhi, Delhi 110007

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Fried and Schmidt[2] have worked on standard convex sublattices. In this paper we generalise concepts of distributive and neutral ideals to convex sublattices. We give criteria for a principal convex sublattice $(a]$ to be a distributive convex sublattice. We extend congruence relation of a lattice L to the lattice of all distributive convex sublattices. Finally, we establish isomorphism between standard convex sublattices and lattices of standard congruences.

1. INTRODUCTION

It is known that every ideal of a lattice L is a convex sablattice of L but the converse is not true. Fried and Schmidt[2] have worked on standard convex sublattices. In Grätzer[1], a problem has been mentioned : 'Generalize the concepts of distributive and neutral ideals to convex sublattices. In this paper we generalise certain results valid for ideals, distributive, standard and neutral ideals to convex sublattices, distributive, standard and neutral convex sublattices respectively.

The concept that 'Every convex sublattice of a lattice L can be expressed uniquely in the form $I \wedge D$ $I \cap D \neq \phi$ where I, is an ideal of L and D is a dual ideal of L, Lemma 6 in (Grätzer[1] p. 19) has helped us a great ideal in solving this problem. We have begun with a few results embedding a lattice L into the lattice C sub (L), lattice of all convex sublattices of L. Theorem 3 develops a criteria for a principal convex sublattice $(a]$ to be a distributive convex sublattice of L. Theorem 4 to Theorem 8 are generalisations of results from ideals, distributive, standard and neutral ideals to convex sublattices, distributive standard and neutral convex sublattices respectively. Theorem 9 enables us to extend a congruence relation of a lattice L to the lattice of distributive convex sublattices of L. Finally Theorem 10 establishes isomorphism of standard convex sublattices and lattices of standard congruences.

In this paper we shall freely use notations and conventions of Grätzer[1]. Necessary definitions and notations have been mentioned at appropriate places.

We begin with the following theorem.

Theorem 1—Let L be a distributive lattice with two comparable elements a and b such that $a \leqslant b \leqslant x \leqslant y$ or $x \leqslant y \leqslant a \leqslant b$ for all $x, y \in L$. Then L has an embedding into C sub (L).

PROOF : Define $\psi : L \to C$ sub (L) by $\psi(x) = [x]\,\theta\,(a, b)$.

$$\psi(x) = \psi(y) \Rightarrow [x]\,\theta\,(a, b) = [y]\,\theta\,(a, b) \Rightarrow x \equiv y\,(\theta\,(a, b)) \Rightarrow x = y$$

by Corollary 5 in Grätzer[1] p. 76. Thus ψ is one-one. It is easy to see that ψ is $\wedge$ and $\vee$-preserving.

Corollary—Every chain of a distributive lattice has an embedding into C sub (L).

The following theorem embeds a lattice L into C sub (L).

Theorem 2—Let ϕ be an injective mapping of a lattice L into L. Define a congruencerelation θ on L as $x \equiv y\,(\theta)$ iff $\phi(x) = \phi(y)$. Then the mapping ψ; $L \to C$ sub (L), defined by $\psi(x) = [x]\,\theta$ is an embedding of L into C sub (L).

Proof is easy.

Observations—(i) Let L be a lattice then for all $a \in L$, $(a]$ and $[a)$ are convex sublattices of L. In fact each ideal of a lattice is a convex sublattice, but not vice versa.

(ii) Every lattice L is contained in C sub (L). (Because each element $a \in L$ could be identified with a convex sublattice $(a] \wedge [a)$, Lemma 6 in Grätzer[1] p. 19.

(iii) Let L^0 be a lattice with zero. Then $P(J_0, J_1, \ldots, J_{n-1})$ is a convex sublattice of L^0 where $J_0, J_1, J_2, \ldots, J_{n-1}$ are ideals, hence are convex sublattices of L^0.

PROOF : As mentioned in the proof of Lemma 8 in (Grätzer[1] p. 29).

$$P(J_0, J_1, \ldots J_n) = \{x \mid 0 \leqslant x \leqslant p(j_0, j_1 \ldots, j_n)$$

$$\text{for some } j_0 \in J_0, j_1 \in J_1, \ldots, j_n \in J_n\}.$$

Let $a, b \in P(J_0, J_1, \ldots, J_n)$ and $x \in L$ be such that $a \leqslant x \leqslant b$ then $0 \leqslant a \leqslant x \leqslant b \leqslant p(j_0, j_1, \ldots, j_n)$ for some $j_0 \in J_0, j_1 \in J_1, \ldots, j_n \in J_n$. Thus $0 \leqslant x \leqslant p(j_0, j_1, \ldots, j_n) \Rightarrow x \in P(J_0, J_1, \ldots, J_n)$.

Q. E. D.

(iv) Let L^0 be a lattice with zero, I be an ideal of L^0 and p and q be n-ary polynomials over L. Let us further assume that for all $a \in I$, there exists $b \in I$ such that $a \leqslant b$ and

$p(b, c_1, c_2, \ldots, c_{n-1}) = q(b, c_1, c_2, \ldots, c_{n-1})$; $c_1, c_2, \ldots, c_{n-1} \in L^0$. Then $p(I, J_1, \ldots, J_{n-1}) = (q\ I, J_1, \ldots, J_{n-1})$ holds is C sub (L^0) for all $J_1, J_2, \ldots, J_{n-1} \in C$ sub (L^0).

(Proof is similar to the proof of Lemma 2 in Grätzer[1] p. 146.)

In the next theorem we develop a criteria for a principal convex sublattice $(a]$ to be a distributive convex sublattice of L.

Theorem 3—$(a]$ is a distributive convex sublattice of L iff the mapping $\phi : I \to (a] \vee I$ is a homomorphism of C sub (L) onto $[(a])$.

PROOF : Let $\phi : I \to (a] \vee I$ be a homomorphism. Then $\phi(I \wedge J) = \phi(I) \wedge \phi(J)$. i.e. $(a] \vee (I \wedge J) = ((a] \vee I) \wedge ((a] \vee J)$ must hold in $[(a])$, which proves that $(a]$ is distributive in C sub L.

Conversely —Assume that $(a]$ is a distributive convex sublattice of C sub (L). Obviously, by definition ϕ is onto.

$$\text{(i)}\quad \phi(I \vee J) = (a] \vee (I \vee J) = ((a] \vee I) \vee (a] \vee J)$$
$$= \phi(I) \vee \phi(J)$$

$$\text{(ii)}\quad \phi(I \wedge J) = (a] \vee (I \wedge J) = ((a] \vee I) \wedge ((a] \vee J) = \phi(I) \wedge \phi(J)$$
$$(\because (a] \text{ is distributive})$$

(i) and (ii) prove that ϕ is a $\vee$ and $\wedge$-preserving map.

Q.E.D.

It is known (p. 19, Grätzer[1])that every convex sublattice I can be written uniquely as $I = P \wedge D$ where P is an ideal and D is a dual ideal. We shall refer to P and D as components of I and set the following definition.

Definition —Call a convex sublattice distributive, standard or neutral iff it's components are respectively distributive, standard or neutral ideals of I (L).

The following theorem is an analogue of a result for ideals in Grätzer[1].

Theorem 4—Let L be a distributive lattice and let I be a convex sublattice of L. Then the following statements are equivalent :

(i) I is distributive

(ii) $\theta(I)$ can be described as

$$x \equiv y\ (\theta(I)) \text{ iff } x \vee i = y \vee i \text{ for some } i \in I.$$

PROOF : To prove (ii) $\Rightarrow$ (i). Let $\theta(I)$ be described as

$$x \equiv y\ (\theta(I)) \text{ iff } x \vee i = y \vee i \text{ for some } i \in I.$$

Let $I = J \wedge D$ and for $i \in I$, $i = j \wedge d$ where $j \in J$ and $d \in D$.

$x \equiv y\,(\theta\,(I))$ iff $x \vee (j \wedge d) = y \vee (j \wedge d)$ for some $j \in J$ and $d \in D$. i. e. $x \equiv y\,(\theta\ (J \wedge D))$ iff $x \vee (j \wedge d) = y \vee (j \wedge d)$ for some $j \in J$ and $d \in D$. $\Rightarrow x \equiv y\,(\theta\,[J] \wedge \theta\,[D])$ iff $x \vee (j \wedge d) = y \vee (j \wedge d)$ for some $j \in J$ and $d \in D$.

$x \equiv y\,(\theta\,[J])$ iff $x \vee (J \wedge d) = y \vee (j \wedge d)$ for some $j \wedge d \in J\, d \wedge D \subset J$. $\Rightarrow J$ is a distributive ideal of L, by Theorem 4 in Grätzer[1] p. 147. Similary, $x \equiv y\,(\theta\,[D])$ iff $x \vee (j \wedge d) = y \vee (j \wedge d)$ for some $j \wedge d \in J \wedge D \subset D$.

$\Rightarrow D$ is a distributive dual ideal of L. $\therefore I = J \wedge D$ is also a distributive ideal of L, which $\Rightarrow I$ is a distributive convex sublattice of L.

(i) $\Rightarrow$ (ii) : Let $I = J \wedge D$ be a distributive convex sublattice of L. $\therefore J$ and D are respectively distributive and dual distributive ideals of L.

$\theta\,(I) = \theta\,(J \wedge D) = \theta\,[J] \wedge \theta\,[D]$ ($\because$ L is distributive)

$x \equiv y\,(\theta\,(I)) \Rightarrow x \equiv y\,(\theta\,[J] \wedge \theta\,[D]) \Rightarrow x \equiv y\,(\theta\,[J])$ and $x \equiv y\,(\theta\,[D])$.

Now, $\therefore J$ is a distributive ideal of L,

$\therefore x \equiv y\,(\theta\,[J]) \Leftrightarrow x \vee j = y \vee j$ for some $j \in J$ and since D is dual distributive ideal of L. $\because x \equiv y\,(\theta\,[D]) \Leftrightarrow x \vee d = y \vee d$ for some $d \in D$. Now, $(x \vee j)(x \vee d) = (y \vee j) \wedge (y \vee d)$ i. e. $x \vee (j \wedge d) = y \vee (j \wedge d)$ $\because$ x and $y \in L$ and L is distributive). $\Rightarrow x \vee i = y \vee i$ for some $i = j \wedge d \in I = J \wedge D$, which proves (i) $\Leftrightarrow$ (ii).

Q. E. D.

Following theorem is generalisation of a result (iv) of Theorem 6 in Grätzer[1] p. 148.

Theorem 5—Let I be a neutral convex sublattice of a lattice L. Then I is distributive, dually distributive and for all J and $K \in C$ sub (L), $I \wedge J = I \wedge K$ and $I \vee J = I \vee K \Rightarrow J = K$.

PROOF : Let $I = J \wedge D$ be a neutral convex sublattice of L. J is neutral ideal and D is a dual neutral ideal of L. Hence J is a distributive ideal of L and so is the dual ideal D of L. $\therefore I$ is distributive convex sublattice of L. Similarly I could be shown to be a dually distributive convex sublattice of L.

Now, let for all $J, K \in C$ sub (L), $I \wedge J = I \wedge K$ and $I \vee J = I \vee K$. For any P and $Q \in I\,(L)$

$$J \wedge P = J \wedge Q \text{ and } J \vee P = J \vee Q \Rightarrow P = Q, \qquad \ldots(1)$$

by (iv) of Theorem 6, (p. 148, Grätzer[1]).

By similar argument for D, we can prove that

$$D \wedge P = D \wedge Q \text{ and } D \vee P = D \vee Q \Rightarrow P = Q \qquad \ldots(2)$$

$\therefore (J \wedge P) \wedge (D \wedge P) = (J \wedge Q) \wedge (D \wedge Q)$.

i.e. $(J \wedge D) \wedge P = (J \wedge D) \wedge Q \Rightarrow I \wedge P = I \wedge Q$ and

$(J \vee P) \vee (D \vee P) = (J \vee Q) \vee (D \vee Q)$

i. e. $(J \vee D) \vee P = (J \vee D) \vee Q \Rightarrow I \vee P = I \vee Q$.

Using (1) and (2), we get

$$I \wedge P = I \wedge Q \text{ and } I \vee P = I \vee Q \Rightarrow P = Q.$$

Q.E.D.

Corollary—For each distributive, standard or neutral convex sublattice I, the map $\psi_I : C \text{ sub } (L) \to C \text{ sub } (L) \times C \text{ sub } (L)$, defined by $\psi_I (J) = < I \wedge J, I \vee J >$ is an isomorphism.

PROOF : Let $\psi_I (J) = \psi_I (K) \Rightarrow < I \wedge J, I \vee J > = < I \wedge K, I \vee K > \Rightarrow I \wedge J = I \wedge K$ and $I \vee J = I \vee K \Rightarrow J = K$, by Theorem 5.

$\therefore \psi_I$ is one-one.

That ψ_I is an onto map, follows from it's definition. Now

$$\begin{aligned}
\psi_I (J \wedge K) &= < I \wedge (J \wedge K), I \vee (J \wedge K) > \\
&= < I \wedge J \wedge K, (I \vee J) \wedge (I \vee K) > (\because I \text{ is distributive}) \\
&= < I \wedge J, I \vee J > \wedge < I \wedge K, I \vee K > \\
&= \psi_I (J) \wedge \psi_I (K).
\end{aligned}$$

Thus ψ_I is $\wedge$-preserving. It remains to prove that

$$\psi_I (J \vee K) = \psi_I (J) \vee \psi_I (K)$$

$$\begin{aligned}
\psi_I (J \vee K) &= < I \wedge (J \vee K), I \vee (J \vee K) > \\
&= < (I \wedge J) \vee (I \wedge K) (I \vee J) \vee (I \vee K) > \\
&\qquad (\because I \text{ is distributive}) \\
&= < I \wedge J, I \vee J > \vee < I \wedge K, I \vee K > \\
&= \psi_I (J) \vee \psi_I (K). \text{ Thus } \psi_I \text{ is } \vee\text{-preserving.}
\end{aligned}$$

Q.E.D.

The next theorem is generalisation of (iii) of Theorem 6 in Grätzer p. 148.

Theorem 6—If I is a neutral convex sublattice of L then for all J and $K \in C$ sub (L); I, J and K generate a distributive convex sublattice.

PROOF : Let $I = P_1 \wedge Q_1$, $J = P_2 \wedge Q_2$ and $K = P_3 \wedge Q_3$. Since I is a a neutral convex sublattice of L. Hence P_1 is a neutral ideal of L and Q_1 is a dually neutral ideal of L. Then by (iii) Theorem 6 in Grätzer[1] (p. 148), P_1, P_2 and P_3 generate a distributive ideal P and Q_1, Q_2 and Q_3 generate a dually distributive ideal Q, say. Then $P \wedge Q$ is the required distributive convex sublattice of L generated by I, J and K. Q.E.D.

Fried and Schmidt[2] have defined, distributive, standard and neutral congruences with the help of distributive, standard or distributive ideals. In the following definition we define distributive, standard and neutral convex congruences.

Definition—(i) Call a congruence relation distributive, standard or neutral convex sublattice iff $\theta = \theta(J)$ for some distributive, standard or neutral convex sublattice J of L.

Recall a definition from Grätzer[1] (p. 149) : Two congruences ϕ and ψ of a lattice are said to permute (or are permutable) iff $\phi \vee \psi = \phi.\ \psi$, where $\phi.\ \psi$ is the binary relation defined by $a \equiv b\ (\phi.\ \psi)$ iff there exists $c \in L$ with $a \equiv c\ (\phi)$ and $c \equiv b\ (\psi)$. An equivalent definition is that ϕ and ψ permute iff $\phi.\ \psi = \psi.\ \phi$, which is, in turn, equivalent to $\phi.\ \psi$ being a congruence relation.

The following result is a generalised form of Theorem 9 in Grätzer[1], p. 149.

Theorem 7—Any two standard convex congruences of distributive lattice permute.

PROOF : Let ϕ and ψ be two standard convex congruences and let $\phi = \theta(I)$ and $\psi = \theta(J)$, I and J are standard convex sublattices of L. Let $I = P \wedge Q$ and $J = P' \wedge Q'$ where P, P' are standard ideals of L and Q, Q' are dual standard ideals of L.

Proving $\phi.\ \psi = \psi.\ \phi$ amounts to proving that if there exists $c \in L$ such that $a \equiv c\ (\phi\ (P \wedge Q))$ and $c \equiv b\ (\psi\ (P' \wedge Q'))$ then there exists $c' \in L$ such that $a \equiv c'\ (\psi\ (P' \wedge Q'))$ and $c' \equiv (b\ (\phi\ (P \wedge Q))$. Now, since $a \equiv c\ (\phi\ (P \wedge Q))$ and $c \equiv b\ (\psi\ (P' \wedge Q'))$. hence

$$a \equiv c\ (\phi\ (P)) \text{ and } a \equiv c\ (\phi\ (Q)). \qquad \ldots(1)$$

(Since $\phi\ (P \wedge Q) = \phi\ (P) \wedge \phi\ (Q)$, because L is a distributive lattice) and $c \equiv b\ (\psi\ (P'))$ and $c \equiv b\ (\psi\ (Q'))$.

Now $\because$ P and Q are standard ideals of L, hence by (Theorem 9, page 149, Gräzer[1]) $a \equiv c\ (\psi\ (P))$ and $c \equiv b\ (\phi\ (P'))$ and $a \equiv c\ (\psi\ (Q))$, $c \equiv b\ (\phi\ (Q')) \Rightarrow a \equiv c\ (\psi\ (P \wedge Q))$ and $c \equiv b\ (\phi\ (P' \wedge Q')) \Rightarrow a \equiv c\ (\psi\ (I))$ and $c \equiv b\ (\psi\ (J))$. Q.E.D.

The following theorem is a generalised form of Theorem 10 in Grätzer[1] p. 149.

Theorem 8—In a sectionally and dually sectionally complemented distributive lattice all congruences are standard convex.

Lemma —Let θ be a congruence relation then Kern θ is a convex sublattice.

PROOF : Let $a \equiv 0\ (\theta)$ and $b \equiv 0\ (\theta)$ and let $c \in L$ be such that $a \leqslant c \leqslant b$. $c \leqslant b \equiv 0\ (\theta) \Rightarrow c \equiv 0\ (\theta)\ \therefore c \in$ Kern θ.

Proof of Theorem—Let $I = [0]\ \theta$ be Kern of θ. By the above lemma, I is a convex sublattice of L. Let $a, b \in L$ and $a \equiv b\ (\theta)$ and let c be the complement of $a \wedge b$ in $[0, a \vee b]$, $c = c \wedge (a \wedge b) \equiv c \wedge (a \wedge b)\ (\theta) = 0\ (\theta)\ \therefore c \in I$.

Thus $a \equiv b\ (\theta)$ iff $(a \wedge b) \vee i = a \vee b$ for some $i \in I$. Hence I is a standard ideal and hence $\theta = \theta\ [I]$.

Again let $=D\ [1]\ \theta$ be convex dual Kern of θ and let c be the dual complement of $a \vee b$ in $[a \wedge b, 1]$.

$c = c \vee (a \wedge b) \equiv c \vee (a \vee b)\ (\theta) = 1\ (\theta)$, Hence $c \in D$. Thus $a \equiv b\ (\theta)$ iff $(a \vee b) \wedge c = a \vee b$ for some $c \in D$, which $\Rightarrow D$ is a dually standard ideal and $\theta = \theta\ [D]$ $\therefore \theta = \theta\ [I] \wedge \theta\ [D] = \theta\ [I \wedge D]$ ($\because L$ is a distributive lattice). Thus $\theta = \theta\ (J)$ where $J = I \wedge D$ with I and D as standard and dually standard ideals of L. Hence, by definition, J is a standard convex sublattice of L, thus proving that θ is standard convex congruence of L.

Q.E.D.

Corollary —A lattice generated by a set of distributive convex sublattices is distributive.

(In view of Ex. 1 (of Grätzer[1], p 140), proof follows on similar lines).

Let C_D sub (L) denote the class of distributive convex sublattices of L.

Following theorem proves the extension of a congruence relation of a lattice to C_D sub (L).

Theorem 9—Consider the lattice L as a sublattice of $I\ (L)$, under the natural embedding $x \to (x]$. Then every congruence relation of L can be extended to C_D sub (L).

PROOF : Let $x \equiv y\ (\theta\ [I]) \Leftarrow\Rightarrow (x] \equiv (y]\ (\theta_I)$ ...(1)

(by proof of Theorem 4 in Grätzer[1] (p. 147).

Now, let $J \in C_D$ sub (L).

$J = I \wedge D$, where I is an ideal of L and D is a dual ideal of $L \in I(L)$ $x \equiv y$ $(\theta [D])$ $\Leftrightarrow\Rightarrow$ $(x] \equiv (y]$ (θ_D) where θ_D is defined as $[x] = (y]$ (θ_D) iff $(x] \vee D = (y] \vee D$.

By (1) and 2 we get

$(x] \equiv (y]$ $(\theta_I \wedge \theta_D)$. Hence $(x] = (y]$ $(\theta_{I \wedge D})$, by proof of Theorem 9 (p 143).

Hence $(x] \equiv (y]$ (θ_J) and θ_J is the required extension of $\theta [I]$ from L to C_D sub (L).

Q.E.D.

Let C_{std} sub (L) denote the class of standard convex sublattices of L.

The next result establishes isomorphism between the lattices of standard convex sublattices of lattices of standard congruences.

Theorem 10—Let L be a distributive lattice and let $J \subseteq L^2$, then $\psi : J \to \theta(J)$ is an isomorphism between the lattice of standard convex sublattices and lattices of standard congruences.

PROOF : Let $\psi(J) = \psi(K)$; $J, K \in C_{\text{std}}$ sub (L) and let $J = I \wedge D$ and $K = I' \wedge D'$, where I and I' are standard ideals of L and D and D' are dual standard ideals of L.

$\theta(J) = \theta(K) \Rightarrow \theta(I \wedge D) = \theta(I' \wedge D')$.

Now, since L is a distributive lattice, hence the above equation reduces to

$$\theta [I] \wedge \theta [D] = \theta [I'] \wedge \theta [D'].$$

$a \equiv b$ $(\theta (J)) \Rightarrow a \equiv b$ $(\theta [I])$ and $a \equiv b$ $(\theta[D])$ and

$a \equiv b$ $(\theta (K)) \Rightarrow a \equiv b$ $(\theta [I'])$ and $a \equiv b$ $(\theta [D'])$.

Thus $a \equiv b$ $(\theta [I])$ and $a \equiv b$ $(\theta [I']) \Rightarrow \theta [I'] = \theta [I'] \Rightarrow I = I'$, by Ex. 15 in Grätzer (p. 150).

Similarly, $a \equiv b$ $(\theta [D])$ and $a \equiv b$ $(\theta [D']) \Rightarrow \theta [D] = \theta [D'] \Rightarrow D = D'$.

Thus $J = I \wedge D = I' \wedge D' = K \therefore \psi$ is one-one.

Also that ψ is an onto map is clear by its definition.

$$\psi (J \wedge K) = \theta (J \wedge K) = \theta (I \wedge D) \wedge ((I' \wedge D'))$$

$$= \theta [I] \wedge \theta [D] \wedge \theta [I'] \wedge \theta [D'] \text{ (By Ex. 15 in Grätzer[1], p. 150)}$$

(equation continued on p. 777)

$= \theta[I \wedge D] \wedge \theta[I' \wedge D'] = \theta(J) \wedge \theta(K) = \psi(J) \wedge \psi(K)$. Thus ψ is $\wedge$-preserving. Again, $\psi(J \vee K) = \theta((I \vee D) \vee (I' \vee D'))$

$$= \theta(I \vee D) \vee \theta(I' \vee D').$$

(This follows by Ex. 15 (in Grätzer[1], p. 150).

$$= \theta(J) \vee \theta(K) = \psi(J) \vee \psi(K).$$

$\therefore$ ψ is $\vee$-preserving.

Q.E.D.

References

1. G. Grätzer, *General Lattice Theory*, Birkhäuser-Verlag, Bessel, 1978.
2. E. Fried, and E. T. Schmidt., Standard sublattices, *Algebra Universalis* 5 (1975), p. 203–11.