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$LH_{a,\mu}$ - SPACES AND DELTA DISTRIBUTION

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It is proved that the conventional Laplace-Hankel transform is a continuous linear mapping from a space of type $LH_{a,\mu}$ into some other space of type $LH_{a,\mu}$. The Laplace-Hankel transform for δ-distribution is obtained.

1. INTRODUCTION

The theory of generalized integral transforms (distributional integral transforms) has its origin in the work of Schwartz[1] who first extended Fourier transform to generalized functions. Gelfand and Shilov[2] introduced spaces of type S and used them to solve Cauchy problem. Lee[3] introduced spaces of type H_μ and used them for the study of Dirichlet problem in cylindrical coordinates.

Motivated from the work of Gelfand and Shilov and Lee, the author (Tiwari[4,5,6]) introduced spaces of type $LH_{a,\mu}$ and discussed its several properties.

In Tiwari[6] Laplace-Hankel transform of members of the dual spaces of type $LH_{a,\mu}$ is defined. Further in Tiwari[6] Laplace-Hankel transform of distributions and impulsive distributions (linear combination of δ distribution and its derivatives) are obtained. In Tiwari[6,7] these spaces are used to solve the equation of propagation of heat in cylindrical coordinates imposing the generalized boundary conditions. Several topological properties of these spaces are proved in Tiwari[8].

We denote by Ω_1 and Ω_2 the open sets $(R_+) \times (R_+)$ and $(R_-) \times (R_+)$ respectively. Further the classes of infinitely differentiable functions defined on Ω_1 and Ω_2 are respectively symbolized by $E(\Omega_1)$ and $E(\Omega_2)$. We use the notation Δ for the operator $x^{-1}\frac{\partial}{\partial x}$. Unless otherwise specified, we choose a, μ, α, β from $\mathbf{R}$ with $\alpha, \beta > 0$ and the letters k, q, l mean 0, 1, 2, ... Also the notation 0° wherever it occurs in the sequel, means the number 1.

1.1. *The Space* $LH_{a,\mu,\alpha,m}$—For given $m > 0$ it is defined as $LH_{a,\mu,\alpha,m} = \{\varphi : \varphi \in E(\Omega_1)$:

$$r^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \right.$$

$$\left. \times \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right| \leqslant C_{q,l,\varepsilon} (m + \delta)^k k^{k\alpha}\}$$

where δ is any number greater than 0 and $C_{q,l,\varepsilon}$ is a constant depending upon δ, and the function φ.

1.2. *The Space* $LH^{\beta,n}_{a,\mu}$ For given $n > 0$, we define

$$LH^{\beta,n}_{a,\mu} = \left\{ \varphi : \varphi \in E(\Omega_1), \right.$$

$$r^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q . \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_{kl\eta} (n + \eta)^q q^{q\beta}\}$$

where the constant $C_{kl\eta}$ depends on the function φ and the number $\eta > 0$.

1.3. *The Space* $LH^{\beta,n}_{a,\mu,\alpha,m}$ —We define the space as

$$LH^{\beta,n}_{a,\mu,\alpha,m} = \{\varphi : \varphi \in E(\Omega_1),$$

$$r^{\mu}_{a,k,q,l} \equiv \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \right.$$

$$\left. \times \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_{\varepsilon\eta} (m + \delta)^k (n + \eta)^q k^{k\alpha} q^{q\beta}\}$$

for given $m, n > 0$ and for $\delta, \eta > 0$. The constant $C_{\varepsilon\eta}$ depends on the function φ.

1.4. *The Space* $LH_{a,\mu,\alpha}$—It is defined as

$$LH_{a,\mu,\alpha} = \{\varphi : \varphi \in E(\Omega_1);$$

$$r^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \right.$$

$$\left. \times \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_{ql} A^k K^{k\alpha}$$

where the constants A and C_{ql} depend on the function φ.

1.5. *The Space* $L^{\vee}H_{a,\mu,\alpha}$—This space is defined as

$$L^{\vee}H_{a,\mu,\alpha} = \varphi : \{\varphi^{\vee}(t, x) = \varphi(-t, x) \in LH_{a,\mu,\alpha};$$

$$\rho^{\mu}_{a,k,q,l}(\varphi) \equiv \sup_{\Omega_2} \Big| x^k e^{-at} D_t^l \Delta^q \times \Big\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \Big\} \Big|$$

$$\leqslant C_{ql} A^k k^{k\alpha}\}$$

Unless specified otherwise, the spaces $LH_{a,\mu,\alpha,m}$, $LH^{\beta,n}_{a,\mu}$, $LH^{\beta,n}_{a,\mu,\alpha,m}$, $LH_{a,\mu,\alpha}$ and $L^{\vee}H_{a,\mu,\alpha}$ will henceforth be considered equipped with their natural Hausdorff locally convex topologies to be denoted respectively by

$$T_{a,\mu,\alpha,m},\ T^{\beta,n}_{a,\mu}T^{\beta,n}_{a,\mu,\alpha,m},\ T_{a,\mu,\alpha} \text{ and } \overset{\vee}{T}_{a,\mu,\alpha}.$$

1.6. *The Space* $\widetilde{LH}_{a,\mu,\alpha}$—It is defined as

$$\widetilde{LH}_{a,\mu,\alpha} = \{\Phi : \Phi = Lh_{\mu}\varphi,\quad \varphi \in LH_{a,\mu,\alpha}\}.$$

where Φ is Laplace-Hankel transform (Tiwari[4], p.1538) of function φ given by

$$\Phi(s, y) = LH_{\mu}\varphi(t, x)$$

$$= \int_0^{\infty}\int_0^{\infty} e^{-st}\sqrt{xy}\, J_{\mu}(x, y)\varphi(t, x)\, dx\, dt.$$

The locally convex topology, denoted by $\widetilde{T}_{a,\mu,\alpha}$ on $\widetilde{LH}_{a,\mu,\alpha}$ is generated by the family of seminorms $\{\lambda_{a,k,q,l}\}^{\ \infty}_{k,q,l=0}$

where

$$\lambda^{\mu}_{a,k,q,l}(\Phi) = r^{\mu}_{a,k,q,l}(\varphi),\ \varphi \in LH_{a,\mu,\alpha}.$$

1.7. *The Space* $\widetilde{L^{\vee}H}_{a,\mu,\alpha}$—It is defined as

$$\widetilde{L^{\vee}H}_{a,\mu,\alpha} = \{\Phi : \Phi = Lh^{-}_{\mu}\varphi,\ \varphi \in L^{\vee}H_{a,\mu,\alpha}\}$$

where $\varphi \in L^{v}H_{a,\mu,\alpha}$ (see Tiwari[4], p. 1536) and

$$\Phi(s,y) = Lh_{\mu}^{-}\{\varphi(t,x)\}$$

$$= \int_{-\infty}^{0}\int_{0}^{\infty} e^{-st}\sqrt{xy}\, J_{\mu}(x,y)\,\varphi(t,x)\,dx\,dt.$$

This space is equipped with the natural Hausdorff locally convex topology $\tilde{T}^{v}_{a,\mu,\alpha}$, generated by the family of seminorms

$$\left\{\sigma^{\mu}_{k,q,l}\right\}_{k,q,l=0}^{\infty} \text{ where } \sigma^{\mu}_{a,k,q,l}(\Phi) = \rho^{\mu}_{a,q,k,l}(\varphi),\ \varphi \in L^{v}H_{a,\mu,\alpha}.$$

2. Conventional Laplace-Hankel Transformation

Following Zemanian[9] we define the following operators

$$N_{\mu}\varphi(t,x) = x^{\left(\frac{2\mu+1}{2}\right)} D_x\left\{x^{-\left(\frac{2\mu+1}{2}\right)}\varphi(t,x)\right\}$$

$$N_{\mu}^{-1}\varphi(t,x) = -x^{\left(\frac{2\mu+1}{2}\right)}\int_{x}^{\infty} v^{-\left(\frac{2\mu+1}{2}\right)}\varphi(t,v)\,dv$$

$$M_{\mu}\varphi(t,x) = x^{-\left(\frac{2\mu+1}{2}\right)} D_x\left\{x^{\left(\frac{2\mu+1}{2}\right)}\varphi(t,x)\right\}.$$

Theorem 2.1—For $\varphi \in LH_{a,\mu,\alpha,m}$: $N_{\mu}\varphi \in LH_{a,\mu+1,\alpha,m}$ and the map N_{μ} : $(LH_{a,\mu,\alpha,m};\ T_{a,\mu,\alpha,m}) \to$

$(LH_{a,\mu+1,\alpha,m};\ T_{a,\mu+1,\alpha,m})$ is one-one linear and continuous :

Proof : Let $\varphi \in LH_{a,\mu,\alpha,m}$. Then

$$\sup_{\Omega_1}\left| x^k e^{at} D_t^l \Delta^q\left\{x^{-\left(\frac{2(\mu+1)+1}{2}\right)} N_{\mu}\varphi\right\}\right|$$

$$= \sup_{\Omega_1} x^k e^{at} D_t^l \Delta^q\left[x^{-1}\, x^{-\left(\frac{2\mu+1}{2}\right)}\left\{x\,\frac{2\mu+1}{2}\right.\right.$$

$$\left.\left. D_x\left(x^{\frac{-2\mu+1}{2}}\varphi\right)\right\}\right]$$

$$= \sup_{\Omega_1} x^k e^{at} D_t^l \Delta^{q+1}\left\{x^{-\left(\frac{2\mu+1}{2}\right)}\varphi\right\}\Bigg|$$

(*equation continued on p.* 804)

$$\leqslant C'_{qls} \quad (m + \delta)^k \, k^{k\alpha}.$$

Thus $N_\mu(\varphi) \in LH_{a,\mu+1,\alpha,m}$ and therefore N_μ is well-defined. It is obviously linear. It is injective, for

$$N_\mu \varphi = 0$$

$$\Rightarrow D_x \left[x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right] = 0$$

$$\Rightarrow x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) = C(t)$$

where $C(t)$ is a constant with respect to x. If $C(t) = 0$, $\varphi = 0$; otherwise

$$\varphi(t, x) = C(t) \, x^{\left(\frac{2\mu+1}{2}\right)}$$

$$\Rightarrow \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right|$$

$$= \sup_{\Omega_1} \left| x^k e^{at} C(t) \right| \quad \text{for } q = l = 0.$$

As the right hand side of the above equation is not bounded, we conclude that $\varphi \notin LH_{a,\mu,\alpha,m}$ which is a contradiction. Hence $C(t) = 0$, that is $\varphi = 0$.

This proves that N_μ is injective. For the continuity of N_μ, observe that

$$r^{\mu+1}_{a,k,q,l}(N_\mu \varphi) \leqslant M \, r^{\mu}_{a,k,q,l}(\varphi)$$

M is some constant.

Similarly we prove :

Theorem 2.2—The operator N_μ^{-1} is a continuous injective map from $(LH_{a,\mu+1,\alpha,m}; T_{a,\mu+1,\alpha,m})$ to $(LH_{a,\mu,\alpha,m}; T_{a,\mu,\alpha,m})$.

Combining the above two theorems, we get :

Theorem 2.3—The operator N_μ (respectively N_μ^{-1}) is a topological isomorphism from the space $LH_{a,\mu,\alpha,m}$ (respectively $LH_{a,\mu+1,\alpha,m}$) onto the space $LH_{a,\mu+1,\alpha,m}$ (respectively $LH_{a,\mu,\alpha,m}$).

Proceeding on the lines of proof of Theorem 2.1 we have

Theorem 2.4 – For $\varphi \in LH_{a,\mu+1,\alpha,m}$, $M_\mu\ \varphi \in LH_{a,\mu,\alpha,m}$ and the map M_μ : $(LH_{a,\mu+1,\alpha,m};\ T_{a,\mu+1\alpha,m}) \to (LH_{a,\mu,\alpha,m};\ T_{a,\mu,\alpha,m})$ is a continuous linear transformation. The proof of the main result of this section exhibit continuity properties of Laplace-Hankel transformation require

Lemma 2.5—For $\varphi \in LH_{a,\mu,\alpha,m}$ and

$\Phi(s, y) = Lh_\mu\ \varphi(t, x)$, we have

(i) $N_{\mu+q+k-1} \cdots N_{\mu+q}\ x^q\ \varphi(t, x)$

$$= x^r\ N_{\mu+k-1} \cdots\ N_\mu\ \varphi(t, x)$$

(ii) $N_{\mu+k-1} \cdots\ N_\mu\ \varphi(t, x)$

$$= x^{\left(\frac{2\mu+2k+1}{2}\right)}\ \Delta^k \left[x^{-\left(\frac{2\mu+1}{2}\right)}\ \varphi(t, x)\right]$$

(iii) $N_{\mu+q-1} \cdots\ N_\mu\ \Phi(s, y)$

$$= y^{\left(\frac{2\mu+2q+1}{2}\right)}\ (y^{-1} D_y)^q \left[y^{-\left(\frac{2\mu+1}{2}\right)}\ \Phi(s, y)\right].$$

PROOF : Follows immediately by induction.

Lemma 2.6—Let $\varphi \in LH_{a,\mu,\alpha,m}$ and $2\mu + 1 \geqslant 0$.

Then

(i) $Lh_{\mu+1}(-x\varphi) = N_\mu\ Lh_\mu\ \varphi(t, x) = N_\mu\ \Phi(s, y)$

(ii) $Lh_{\mu+1}(N_\mu\ \varphi) = -y\ Lh_\mu\ \varphi.$

PROOF : (i) Consider

$N_\mu\ \Phi(s, y)$

$$= y^{\left(\frac{2\mu+1}{2}\right)}\ Dy\ y^{-\left(\frac{2\mu+1}{2}\right)} \int_0^\infty \int_0^\infty e^{-st} \sqrt{xy}\ J_\mu(xy)\ \varphi(t, x)\ dx\ dt$$

$$= y^{\left(\frac{2\mu+1}{2}\right)} \int_0^\infty \int_0^\infty e^{-st} \sqrt{x}\ \varphi(t, x)\ Dy\ [y^{-\mu} J_\mu(x y)]\ dx\ dt$$

(equation continued on p. 806)

$$= y\left(\frac{2\mu+1}{2}\right)\int_0^\infty\int_0^\infty e^{-st}\sqrt{x}\,\varphi(t,x)\,[-xy^{-\mu}J_{\mu+1}(xy)]\,dx\,dt$$

since $Dy\,y^{-\mu}J_{\mu(xy)} = -xy^{-\mu}J_{\mu+1}(xy)$, (see Jahnke *et al.*[10] p 154). Hence $N_\mu^o\,\Phi(s,y) = Lh_{\mu+1}(-x\varphi)$ and (i) follows.

(ii) Consider

$$Lh_{\mu+1}(N_u\,\varphi)$$

$$= \int_0^\infty\int_0^\infty e^{-st}\sqrt{xy}\,J_{\mu+1}(xy)\,\{N_\mu\,\varphi(t,x)\}\,dx\,dt$$

$$= \int_0^\infty\int_0^\infty e^{-st}\sqrt{xy}\,J_{\mu+1}(xy)\,(x)^{\frac{2\mu+1}{2}}$$

$$D_x\left(x^{-\left(\frac{2\mu+1}{2}\right)}\varphi\right)\,dx\,dt$$

$$= \int_0^\infty\int_0^\infty \sqrt{y}\;e^{-st}\,\{x^{\mu+1}J_{\mu+1}(xy)\}\left\{D_x(x)^{-\left(\frac{2\mu+1}{2}\right)}\varphi(t,x)\right\}\,dx\,dt$$

$$= -y\,Lh_\mu\,\varphi(t,x)$$

by integrating by parts and using the results

$D_x[x^{\mu+1}J_{\mu+1}(xy)] = yx^{\mu+1}J_\mu(xx)$ of Jahnke *et al.*[10], p. 154.

Lemma 2.7—Consider $\Phi(s,y)$ for $\varphi \in LH_{a,\mu,\alpha,m}$.

Then $(-y)^k N_{\mu+q-1}\ldots N_\mu\,\Phi(s,y)$

$$= Lh_{\mu+k+q}[(-x)^q N_{\mu+k-1}\cdots N_\mu\,\varphi(t,x)].$$

PROOF : From Lemma 2.6 (i), $N_\mu^i\,\Phi = LH_{\mu+1}(-x\varphi)$.

Therefore if $\psi = -x\varphi$, then

$$N_{\mu+1}(N_\mu\,\Phi) = N_{\mu+1}(LH_{\mu+1}\psi) = LH_{\mu+2}(-x\psi)$$

again using Lemma 2.6 (i)

Repeating this process q times and using Lemma 2.6 (i) we get

$$N_{\mu+q-1}\ldots\ N_\mu\ \Phi = Lh_{\mu+q}\ \{(-x)^q\ \varphi\}.$$

Multiplying both sides by $-y$ and writing $\psi = (-x)^q\ \varphi$, we have

$$(-y)\ [N_{\mu+q-1}\ldots N_\mu\ \Phi] = -y\ Lh_{\mu+q}\ (\psi)$$

$$= Lh_{\mu+q+1}\ N_{\mu+q}\ (\varphi)$$

by Lemma 2.6 (ii). Hence

$$(-y)\ [N_{\mu+q+1}\ldots\ N_\mu\Phi] = Lh_{\mu+q+1}\ N_{\mu+q}\ [(-x)^q\varphi].$$

Again repeating this k times and using Lemma 2.6 (ii) we get

$$(-y)^k\ [N_{\mu+q-1}\ldots\ N_\mu\ \Phi]$$

$$= Lh_{\mu+q+k}\ [N_{\mu+k+q-1}\ \ldots N_{\mu+q}\ \ \{(-x)^q)\varphi\}]$$

$$= Lh_{\mu+q+k}\ [(-x)^q\ N_{\mu+k-1}\ \ldots N_\mu\ \varphi].$$

By Lemma 2.5 (i). This completes our proof. We now prove the main result contained in :

Theorem 2.8—Let a, s be two real numbers with $a > 0$, $s > 0$ and $s + a > 0$. Then the conventional Laplace-Hankel transform Lh_μ from

$$\left(LH^{\beta,n}_{a,\mu,\alpha,m}\ ;\ T^{\beta,n}_{a,\mu,\alpha,m} \right) \text{ into } \left(LH^{2\alpha,\ (2e)^{2\alpha}m^2}_{a,\mu,\mu+\beta,e^{2\alpha}mn}\ ; \right.$$

$$\left. T^{2\alpha,(2e)^{2\alpha}m^2}_{a,\mu,\alpha+\beta,e^{2\alpha}mn} \right) \text{ is a continuous linear mapping.}$$

PROOF : For the well defined character of the map Lh_μ, we first prove that

$$\Phi \in LH^{2\alpha,(2e)^{2\alpha}m^2}_{a,\mu,\alpha+\beta,e^{2\alpha}mn} \text{ if } \varphi \in LH^{\beta,n}_{a,\mu,\alpha,m}.$$

Consider

$$E = \left| e^{as}\ D^l_s\ y^k\ (y^{-1}\ D_y)^q \quad \left\{ y^{-\left(\frac{2\mu+1}{2}\right)}\ \Phi\ (s, y) \right\} \right|$$

$$= \left| e^{as}\ D^l_s\ \ y^{-(\mu+q+1/2)}\ (-y)^k\ \{N_{\mu+q-1}\ldots\ N_\mu\ \Phi\} \right|$$

(cf. Lemma (2.5 (iii))

$$= \left| e^{s} D_s^l \; y^{-(\mu+q+1/2)} \, Lh_{\mu+k+q} \{(-x)^q \, N_{\mu+k-1} \ldots N_\mu \varphi\} \right|$$

(cf. Lemma 2.7)

$$= \left| e^{as} \; D_s^l \; y^{-(\mu+q+1/2)} \int_0^\infty \int_0^\infty e^{-st} \, (-x)^q \; x^{\mu+k+1/2} \right.$$

$$\left. \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \sqrt{xy} \; J_{\mu+k+q}(xy) \, dx \, dt \right|$$

by Lemma 2.5 (ii). Since $\left| \frac{J_{\mu+k+q}(xy)}{(xy)^{\mu+q}} \right| \leqslant M_1$, for some constant $M_1 > 0$, see Jahnke *et al.*[10] p. 134, we get

$$E \leqslant M_1 \int_0^\infty \int_0^\infty \left| e^{as} \; D_s^l \, e^{-st} \; x^{2\mu+2q+k+1} \; \Delta^k \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \right\} \right| dx \, dt.$$

Assuming n to be an integer no less than $2\mu + 1$, we have $x^{2\mu+1} < (1 + x^2)^n$, $x > 0$. Hence

$$E \leqslant M_1 \int_0^\infty \int_0^\infty \left| e^{as-st} \; (t)^l \left\{ D_t^{-l} \, e^{-at} \; e^{at} \, D_t^l \right\} \right.$$

$$\left. x^{2q+k} . \; (1 + x^2)^{n+1} \frac{1}{(1 + x^2)} \, \Delta^k . \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \right\} \right| dx \, dt$$

$$\leqslant M_1 \int_0^\infty \int_0^\infty \left| \left\{ e^{as-st} \; t^l \; \frac{e^{-at}}{o^l} \; \frac{1}{1 + x^2} \right\} \right.$$

$$\left. \times \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} x^{2\nu+2q+k} \, e^{at} \; D_t^l \, \Delta^k \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \right\} \right| dx \, dt$$

$$\leqslant M_1 \int_0^\infty \int_0^\infty \left| \frac{e^{as}}{a^l} \; t^l \, e^{-(s+a)t} \right| \frac{dx}{1 + x^2} \, dt$$

$$\times \sum_{\nu=0}^{n+1} \binom{n+1}{\nu} \; r^{\mu}_{a,2\nu+2q+k,k+1} \quad (\varphi) \qquad \ldots(2.1)$$

because $\varphi \in LH^{\beta,n}_{a,\beta,\alpha,m}$.

Let us estimate

$$r^{\mu}_{a,2v+2q+k,k,l}(\varphi) \leqslant C_{\varepsilon\eta}(m+\delta)^{2v+2q+k}(n+\eta)^{k}(2v+2q+k)^{(2v+k+2q)\alpha} \times k^{k\beta}$$

$$\leqslant C_{\varepsilon\eta}(m+\delta)^{2v}(m+\delta)^{2q}(m+\delta)^{k}(n+\eta)^{k}k^{k\beta} \times (2v+k+2q)^{(2v+k)\alpha}.(2v+2q+k)^{2q\alpha}$$

$$\leqslant C_{\varepsilon\eta}.C_1.(m^2+\delta_1)^{q}(mn+\delta_2)^{k}k^{k\beta}.(2v+k)^{(2v+k)\alpha} \times \left[1+\frac{2q}{(2v+k)}\right]^{(2v+k)\alpha}(2q)^{2q\alpha}\left[1+\frac{k+2v}{2q}\right]^{2q\alpha}$$

$\delta_1, \delta_2 > 0$ and C_1 is some constant

$$\leqslant C_{\varepsilon\eta}.C_1(m^2+\delta_1)^{q}(mn+\delta_2)^{k}k^{k\beta}(2v+k)^{2v\alpha} \times (2v+k)^{k\alpha}e^{2q\alpha}(2q)^{2q\alpha}e^{(k+2v)\alpha}$$

$$\leqslant C_{\varepsilon\eta}.C_1(m^2+\delta_1)^{q}e^{2q\alpha}(mn+\delta_2)^{k}k^{k\beta}(2v)^{2v\alpha}\left[1+\frac{k}{2v}\right]^{2v\alpha} \times k^{k\alpha}\left[1+\frac{2v}{k}\right]^{k\alpha}(2q)^{2q\alpha}e^{k\alpha}e^{2v\alpha}$$

$$\leqslant C_{\varepsilon\eta}C_1(e^{2\alpha}m^2+\delta_3)^{q}.(mn+\delta_2)^{k}k^{k(\alpha+\beta)}(2v)^{2v\alpha} \times e_k^{\alpha}e^{(2v)\alpha}(2q)^{2q\alpha}e^{k\alpha}e^{2v\alpha},\ \delta_3 > 0$$

$$\leqslant C'_{\varepsilon\eta}\{(2e)^{2\alpha}m^2+\delta_4\}^{q}\{(e^{2\alpha}mn+\delta_5)\}^{k}k^{k(\alpha+\beta)}q^{2q\alpha}$$

$\delta_4, \delta_5 > 0$ and $C'_{\varepsilon\eta}$ is a constant.

From above and eqn. (2.1):

$\Phi \in LH^{2\alpha,\ (2e)^{2\alpha}m^2}_{a,\mu,\alpha+\beta,e^{2\alpha}m_n}$ and, therefore,

the mapping $Lh_\mu : \varphi \to \Phi$ is well defined. It is obviously linear. Its continuity is a consequence of equation (2.1).

3. The Distributional Laplace-Hankel Transform

Before defining the distributional Laplace-Hankel transform we prove the following theorems.

Theorem 3.1—The mapping R defined by $R(\varphi) = \overset{\vee}{\varphi}$ is a topological isomorphism from the space $(L^{\vee}H_{a,\mu,\alpha};\ \overset{\vee}{T}_{a,\mu,\alpha})$ onto the space $(LH_{a,\mu,\alpha};\ T_{a,\mu,\alpha})$.

Proof : Proof is immediate from the definition of the spaces $LH_{a,\mu,\alpha}$ and $L^{\vee}H_{a,\mu,\alpha}$.

Theorem 3.2—The mapping P : $(\widetilde{LH}_{a,\mu,\alpha};\ \widetilde{T}_{a,\mu,\alpha}) \to (\overset{\vee}{\widetilde{LH}}_{a,\mu,\alpha};\ \overset{\vee}{\widetilde{T}}_{a,\mu,\alpha})$ defined by $P(\Phi) = \overset{\vee}{\Phi}$ is a topological isomorphism for Re $(s) \in (-a, a)$.

Proof : Since the existence of the integral

$$\int_0^{\infty}\int_0^{\infty} \exp(-st)\ \sqrt{xy}\ J_{\mu}(xy)\ \varphi(t, x)\ dx\ dt$$ is equivalent to the existence of the integral

$$\int_{-\infty}^{0}\int_0^{\infty} \exp(-st)\ \sqrt{xy}\ J_{\mu}(xy)\ \varphi(-t,x)\ dx\ dt,$$ the mapping is well defined. Clearly, P is linear, one-one and onto. The continuity of P follows from

$$\sigma^{\mu}_{a',k',q',l}(P\Phi) = \rho^{\mu}_{a',k',q,l}(\overset{\vee}{\Phi})$$

$$= \gamma^{\mu}_{a',k',q,l}(\varphi)$$

$$= \lambda^{\mu}_{a',k',q',l}(\Phi).$$

Since the inverse mapping P^{-1} of P is given by $P^{-1}(\overset{\vee}{\Phi}) = \Phi$, the result is established.

From the above theorems we now define distributional Laplace-Hankel transform LH^{*}_{μ} by

$$\langle Lh^{*}_{\mu} f, \Phi \rangle = 2\pi i \langle f, \overset{\vee}{\Phi} \rangle \qquad \ldots(3.1)$$

where $f \in LH^*_{a',\mu',\alpha}$, $\overset{v}{\varphi} \in LH_{a,\mu,\alpha}$, $\Phi = Lh_\mu\, \varphi$ and $LH^*_{a,\mu,\alpha}$ is the topological dual of the space $LH_{a,\mu,\alpha}$.

We now find Laplace-Hankel transform of δ-distribution where the distribution $\delta_{a,b}$ is defined by

$$<\delta_{a,b},\ \varphi\ (t,x)> = \varphi\ (a,\ b);\ a,\ b > 0.$$

Theorem 3.3—Let $a,\ b > 0$ and $G\ (s,\ y) = \sqrt{by}\ J_\mu\ (by) \exp\ (-sa)$.

Then

$$LH^*_\mu\ \{\delta_{a,b}\} = T_G$$

where T_G is a distribution generated by the function $G\ (s,\ y)$.

PROOF : From the equation (3.1), we get

$$< Lh^*_\mu\ \delta_{a,b}, \Phi> = 2\pi\ i\ \ <\delta_{a,b},\ \overset{v}{\varphi}> = 2\pi i\ \ \overset{v}{\varphi}\ (a,b)$$

$$= 2\pi\ i.\ \frac{1}{2\pi i} \int_0^\infty \sqrt{by}\ J_\mu\ (by)$$

$$\times \int_{\sigma-i\,\infty}^{\sigma+i\infty} \{e^{-as}\ \Phi\ (s,\ y)\}\ ds\ dy$$

using the inverse Laplace-Hankel transform viz.

$$\varphi\ (t,\ x) = \frac{1}{2\pi i} \int^\infty \sqrt{xy}\ J_\mu(xy) \int_{\sigma-i\,\infty}^{\sigma+i\infty} e^{st}\ \Phi\ (s,\ y)\ ds\ dy.$$

Hence

$$< Lh^*_\mu\ \delta_{a,b}\ ,\ \Phi> = \int_0^\infty \int_{\sigma-i\infty}^{\sigma+i\infty} \sqrt{by}\ J_\mu\ (by) \exp\ (-sa)$$

$$\times\ \Phi\ (s,\ y)\ ds\ dy$$

(*equation continued on p.* 812)

$$= < T \sqrt{by}\, J_\mu (by) \exp(-sa), \Phi >$$

$$= <T_G, \Phi>.$$

Consequently $LH^*_\mu\ \delta_{a,b} = T_G$ and our proof is complete.

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