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A FINSLERIAN EXTENSION OF THE GRAVITATIONAL FIELD

SATOSHI IKEDA

Department of Mechanical Engineering, Faculty of Science and Technology, Science University of Tokyo, Noda Chiba 278, Japan

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A structural generalization of the gravitational field is attempted with reference to the theory of Finsler spaces : The vector y is attached to each point x as the internal variable and the new Finsler metric $g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y)$ is constructed by unifying Riemann metric $\gamma_{\lambda\kappa}(x)$ of the external (x)-field spanned by points $\{x\}$ and the Riemann metric $h_{ij}(y)$ of the internal (y)-field spanned by vectors $\{y\}$. The metrical Finsler connection with respect to $g_{\lambda\kappa}(x, y)$ (i. e., $D g_{\lambda\kappa} = 0$) is determined by taking account of the intrinsic behavior of y (i. e., δy^i).

1. INTRODUCTION

As is well known, several kinds of structural extensions of Einstein's gravitational field[1] have been proposed : For example, Brans-Dicke theory[2], Einstein-Cartan theory[3], etc. In those theories, such "non"-Riemannian quantities as scalar, torsion, etc. are introduced, besides the Riemann metric $\gamma_{\lambda\kappa}(x)$ ($\kappa, \lambda = 1, 2, 3, 4$), and the spatial structures become, of course, "non"-Riemannian. However, these resulting "non"-Riemannian gravitational fields may be regarded as "local" in the sense of Yukawa's nonlocal field theory[4], because only the point x ($= x^\kappa$; $\kappa = 1, 2, 3, 4$) is adopted as the independent variable. Therefore, if some new independent (internal) variable is attached to each point, then a new "non"-Riemannian and "nonlocal" gravitational field can be obtained[5]. So, along this line, we shall consider, in this paper. a Finslerian extension of the gravitational field by choosing the vector y as such independent variable.

In section 2, it is shown that our Finsler metric $g_{\lambda\kappa}(x, y)$ of our Finslerian gravitational field is given by (2.4), which is constructed by unifying the external Riemann metric $\gamma_{\lambda\kappa}(x)$ of the external (x)-field spanned by points $\{x\}$ and the internal Riemann metric $h_{ij}(y)$ of the internal (y)-field spanned by vectors $\{y\}$. The metrical Finsler connection D for $g_{\lambda\kappa}$ of (2.4) (i.e., $Dg_{\lambda\kappa} = 0$) is determined by taking account of the intrinsic behavior of the internal vector y in section 3. This process must be performed in order to clarify the whole spatial structure of our Finslerian field.

2. Finslerian Structure I

As mentioned above, the Finslerian "nonlocalization" in our sense is realized by annexing the vector y to each point x as the internal variable[5]. Then, we may consider that there appear two fields around x: One is the external (x)-field spanned by points $\{x\}$ and the other is the internal (y)-field spanned by vectors $\{y\}$. The former is nothing else than the Einstein's gravitational field endowed with the (four-dimensional) Riemannian structure, while the latter may be compared to the so-called internal space associated with each point, which may be assumed to have a (four-dimensional) Riemannian structure, in general. (The latter may be reduced to a Minkowskian structure, if necessary. See below.)

So for our purpose in this paper, we shall first consider a unification between the (x)-and (y)-fields at the stage of metric. That is to say, we shall construct a Finslerian metric $g_{\lambda\kappa}(x, y)$ by unifying the Riemann metric $\gamma_{\lambda\kappa}(x)$ of the (x)-field and the Riemann metric $h_{ij}(y)$ of the (y)-field. (If the (y)-field is flat, $h_{ij}(y)$ reduces to the Minkowski metric). It should be remarked that in order to distinguish the physical function explicity. Greek indices $\kappa, \lambda, \mu, \ldots$ $(= 1, 2, 3, 4)$ are used for external quantities such as y^κ, $g_{\lambda\kappa}$, etc., while Latin indices i, j, k, $(= 1, 2, 3, 4)$ are used for the internal quantities such as y^i, h_{ij}, etc.

Now, if the (y)-field is regarded as the (standard) fibre at the point x, then our unified field may be compared to the vector bundle over the base (x)-field. Then, if the (adapted) dual bases $(dx^\kappa, \delta y^i)$ are properly set, then the metrical structure of the vector bundle is represented by the following square of the arc length :

$$dS^2 = f_{\lambda\kappa}(x, y)\, dx^\kappa\, dx^\lambda + f_{ij}(x, y)\, \delta y^i\, \delta y^j \qquad \ldots(2.1)$$

where the metric tensors $f_{\lambda\kappa}$ and f_{ij} are introduced formally. In (2.1) δy^i is defined by

$$\delta y^i = dy^i + N^i_\mu\, dx^\mu \qquad \ldots(2.2)$$

where the quantity N^i_μ is called the nonlinear connection[7], which physically prescribes the interactions between the (x)-and (y)-fields. (In this case, the (adapted) bases are given[6,7] by $\left(\dfrac{\partial}{\partial x^\mu} - N^i_\mu \dfrac{\partial}{\partial y^i},\ \dfrac{\partial}{\partial y^i} \right)$.

The connection δ represents geometrically the intrinsic connection of y, which differs from the total metrical connection D, and characterizes physically the intrinsic behavior of y, (see section 3). However, from this standpoint, there cannot be considered any relation between $f_{\lambda\kappa}$ and f_{ij} and there cannot be considered any relation between $f_{\lambda\kappa}$ and $\gamma_{\lambda\kappa}$ or h_{ij}. (Of course, the dimension number of the total

space of the vector bundle is 8, not 4.) Therefore the unification in terms of vector bundles is not suitable for our purpose in this paper.

So, we shall reconsider our unification as follows : Within the framework of the theory of Finsler spaces[7] the tangent space at the point x (=fixed) is a (four-dimensional Riemannian space spanned by tangent vectors such as $\{y\}$ and is governed by its Riemann metric such as $h_{ij}(y)$, where the system (i) is chosen properly. And it is known[7] that a Finsler metric such as $h_{\lambda\kappa}(x, y)$ can be made locally not to depend on x (i.e., $h_{\lambda\kappa}(x, y) \to h_{ij}(y)$) under suitable conditions, where the system (κ) is a general one and the system (i) must be chosen properly. Therefore, taking account of these facts, we may consider that our system (i) with respect to y^i and $h_{ij}(y)$ is likened to the above-mentioned system (i), although the (y)-field is not necessarily regarded as the tangent space.

Then, in order to unify the internal (y)-field with the external (x)-field, we shall here introduce the following mapping relation :

$$y^\kappa = e^\kappa_i(x)\, y^i$$

$$h_{\lambda\kappa}(x, y) = e^i_\lambda(x)\, e^j_\kappa(x)\, h_{ij}(y) \qquad \ldots(2.3)$$

where the quantity e denotes the mapping operator of the (y)-field on the (x)-field. By (2.3), the internal quantities y^i and h_{ij} are brought to the external quantities y^κ and $h_{\lambda\kappa}$. The operator e can change the indices $i, j, k\ldots$ into $\kappa, \lambda, \mu, \ldots$, but cannot be regarded as the coordinate transformation matrix between the (i)-and (κ)-systems, because $\mathbb{C}$ cannot combine h_{ij} and $g_{\lambda\kappa}$ [see (2.4)]. [If the (y)-field is flat, then $\mathbb{C}$ becomes a function of (x, y), in order to introduce the Finsler metric $h_{\lambda\kappa}(x, y)$ by means of (2.3)] As is understood from the above, since the (y)-field is embedded in the (x)-field, the mapping process (2.3) may be consided our unification process of the (x)-and (y)-fields.

Now, as to the unified Finsler metric $g_{\lambda\kappa}(x, y)$ of the unified Finslerian gravitational field, it must be constructed by unifying the Riemann metric $\gamma_{\lambda\kappa}(x)$ and the induced Finsler metric $h_{\lambda\kappa}(x,y)$, because $\gamma_{\lambda\kappa}(x)$ dominates the (x)-field from the beginning. At this stage, we can consider several kinds of unifications of $\gamma_{\lambda\kappa}(x)$ and $h_{\lambda\kappa}(x, y)$. But we shall here propose, as the most simple and typical example, the following form :

$$g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y). \qquad \ldots(2.4)$$

Thus, one concrete example of Finsler metric has been obtained for the gravitational field.

Starting from (2.4), we can determine our metrical Finsler connection with respect to $g_{\lambda\kappa}$ (i. e., $D\,g_{\lambda\kappa} = 0$) and can clarify the spatial structure of our Finslerian gravitational field, as will be actually be done in the next Section.

3. Finslerian Structure

As is understood from a physical viewpoint, the vector y shows, as the internal variable, its own intrinsic behavior, which is geometrically grasped by its own intrinsic connection or parallelism (i. e., $\delta\, y^i$) in the (y)-field. As a typical example of the intrinsic behavior of y, we shall here consider the rotational property such as, as Asanov's K-group[8].

$$y^i = K^i_j\ (x)\, y^j\ (\equiv\ y^i_0\ + dy^i) \qquad \ldots(3.1)$$

where $K^i_j\ \left(\equiv K^i_j\ (0) + \dfrac{\partial\, K^i_j}{\partial x^\mu}\, d\,x^\mu;\ y^i_0 \equiv\ K^i_j\ (0)\, y^j\right)$ means the rotation matrix. (K^i_j may be generalized to a function of (x, y), but it is needless for our purpose in this paper. If the (y)-field is flat, K^i_j is reduced to a constant matrix as in the global Lorentz transformation.) From (3.1), we can formally formulate, corresponding to (2.2) the intrinsic parallelism of y (i. e., $\delta\, y$) in the form

$$\delta y^i = dy^i\ +\ N^i_\mu\, dx^\mu\ (=\ 0) \qquad \ldots(3.2)$$

where in this case, N^i_μ is defined by $N^i_\mu\ =\ -\ \dfrac{\partial K^i_j}{\partial x^\mu}\, y^j\ \left(\equiv\ N^i_{j\mu}\, y^j\right)$. (Of course, if the (y)-field is flat, (3.2) reduces to $\delta\, y^i = dy^i\ (=0)$, because $N^i_\mu\ =\ 0$). The condition $\delta\, y^i = 0$ means that one direction is preferred in the (y)-field, so that there arises the anistropic property in the (y)-field.

Since our unification process is realized by the mapping process (2.3), (3.2) is brought to the external (x)-field as follows :

$$\delta\, y^\kappa = e^\kappa_i\ (x)\, \delta y^i = dy^\kappa + N^\kappa_\mu\, d\,x^\mu$$

$$\equiv\ d\,y^\kappa\ +\ N^\kappa_{\mu\lambda}\, y^\lambda\, dx^\mu\ (=\ 0) \qquad \ldots(3.3)$$

where we have put $N^\kappa_\mu = e^\kappa_i\ N^i_\mu - \dfrac{\partial\, e^\kappa_i}{\partial x^\mu}\ e^i_\lambda\ y^\lambda$ and

$N^{\kappa}_{\lambda\mu} = e^{\kappa}_{i}\ e^{j}_{\lambda}\ N^{i}_{j\mu} - \dfrac{\partial\, e^{\kappa}_{i}}{\partial x^{\mu}}\, e^{i}_{\lambda}$. In the case of the (y)-field being flat, since $\mathbb{C}$ becomes a function of (x, y) and $N^{i}_{\mu} = 0$, (3.3) reduces to

$$\delta y^{\kappa} = e^{\kappa}_{i}(x)\,\delta y^{i} = dy^{\kappa} - \frac{\partial\, e^{\kappa}_{i}}{\partial x^{\mu}}\, y^{i}\, dx^{\mu} - \frac{\partial e^{\kappa}_{i}}{\partial y^{\kappa}}\, y^{i}\, dy^{\kappa}.$$

The last term of the right-hand-side is equal to

$$-\frac{\partial e^{\kappa}_{i}}{\partial y^{\kappa}}\, e^{i}_{\lambda}\, y^{\lambda}\,\delta y^{\kappa} = -\frac{\partial\, e^{\kappa}_{i}}{\partial y^{\kappa}}\, e^{i}_{\lambda}\ e^{\kappa}_{\mu}\, y^{\lambda}\,\delta y^{\mu}.$$

Now, it may be considered that δy^{κ} given by (3.3) reflects in the unified field the intrinsic behavior of y^{i}, so that the whole Finslerian structure at the stage of connection is also influenced by δx^{κ} itself. Under these situations, the metrical Finsler connection D with respect to $g_{\lambda\kappa}$ of (2.4) (i. e., $D\, g_{\lambda\kappa} = 0$) can be represented by, for an arbitrary vector X^{μ},

$$DX^{\kappa} = dX^{\kappa} + \Gamma^{\kappa}_{\lambda\mu} X^{\lambda}\, dx^{\mu} + C^{\kappa}_{\lambda\mu} X^{\lambda}\, dy^{\mu}$$

$$= dX^{\kappa} + F^{\kappa}_{\lambda\mu}\ X^{\lambda}\, dx^{\mu} + C^{\kappa}_{\lambda\mu} X^{\lambda}\,\delta\, g^{\mu} \qquad \ldots(3.4)$$

where $F^{\kappa}_{\lambda\mu}$ ($\equiv \Gamma^{\kappa}_{\lambda\mu} - N^{\nu}_{\mu}\ C^{\kappa}_{\lambda\nu}$; see (3.3)) and $C^{\kappa}_{\lambda\mu}$ denote the unified Finslerian coefficients of our unified field. Those coefficients will be related to the intrinsic behaviour of y, as will be shown in the following. By the way, the horizontal coefficient connection $F^{\kappa}_{\lambda\mu}$represnts the concept of unified gauge field[9] for the Finslerian gravitational field.

In order to obtain the relation between the two connections D and δ, we shall consider as follows : First, since $\delta y^{\kappa} \neq Dy^{\kappa}$ in our case, there hold good the relations ($D\, g_{\lambda\kappa} = 0$ and $Dh_{\lambda\kappa} \neq 0$) and ($\delta\, g_{\lambda\kappa} \neq 0$ & $\delta h_{\lambda\theta} = 0$). (The metrical conditions $\delta\, h_{ij} = 0$ are assumed under $\delta y^{i} = 0$ so that $\delta h_{\lambda\kappa} = 0$ are assumed under the absolute parallelism of $\mathbb{C}$, i. e., $\delta\, \mathbb{C} = 0$.) Next, as has already been mentioned in section 3 of Ikeda[10], it may be reconsidered from the relation ($Dg_{\lambda\kappa} = 0$ and $\delta\ g_{\lambda\kappa} \neq 0$) that the connection D is a metrical connection for $g_{\lambda\kappa}$ derived from the non-metrical one δ.

Thirdly, using Kawaguchi's theorem[11] which supplies a method to make a non-metrical connection metrical, we obtain as follows (neglecting arbitrarines) :

$$D\,y^{\kappa} = \delta\, y^{\kappa} + \tfrac{1}{2}\, g^{\kappa\nu}\,(\delta\, g_{\nu\lambda})\, y^{\lambda}. \qquad \ldots(3.5)$$

Therefore, from (3.5), the following relations can be obtained using (3.3) and (3.4) :

$$\Gamma^{\kappa}_{\lambda\mu} = N^{\kappa}_{\lambda\mu} + \frac{1}{2}\, g^{\kappa\nu}\left(\frac{\partial\, g_{\nu\lambda}}{\partial\, x^{\mu}} - N^{l}_{\gamma\mu}\, g_{l\lambda} - N^{l}_{\lambda\mu}\, g_{\nu l}\right). \qquad \ldots(3.6)$$

$F^{\kappa}_{\lambda\mu}$ can also determined by using (3.6). As to the coefficient $C^{\kappa}_{\lambda\mu}$, it is also given by (3.5), if the vertical coefficient of connection appears in δy^{κ} (3.3), as in the case of the flat (y)-field. At this final stage, we may say that the intrinsic behavior of y^{i} or y^{κ} (i.e. $\delta\, y^{i}$ or δy^{κ}) represented by N^{l}_{μ} (3.2) or N^{κ}_{μ}, $N^{\kappa}_{\lambda\mu}$ (3.3) is absorbed into $\Gamma^{\kappa}_{\lambda\mu}$, $F^{\kappa}_{\lambda\mu}$ (3.4) by means of the relation (3.5) or (3.6).

Thus, the spatial structure of our Finslerian gravitational field, especially the c on-nection structure based on the Finsler metric (2.4), has been completely clarified by taking into account the intrinsic behavior of the internal variable y.

References

1. A. Einstein, *Ann. der. Phys.* **49** (1916), 769.
2. C. Brans, and R. H. Dicke, *Phys. Rev.* 124 (1961), 925.
3. F. Hehl, P. van der Heyde, G. Kerlick, and J. Nester, *Rev.Mo d. Phys.* 48 (1976) 393.
4. H. Yukawa, *Phys. Rev.* 77 (1950) 219.
5. S. Ikeda, *Lett. Nuovo Cimmento* 21 (1978), 165.
6. R. Miron. In "*Proc 2nd Nat. Sem. Finsler spaces*" Brasov Univ., 1983, pp. 147-88.
7. M. Matsumoto, *The Theory of Finsler Connections*. Okayama Univ., 1977 ; *Foundations of Finsler Geometry and Special Finsler Spaces* (Unpublished, 1977).
8. G. S. Asanov, *Ann. der Phys.* 34 (1977), 169.
9. D. Ivanenko, in "*Relativity, Quanta and Cosmology*" (ed. M. Pantaleo) Jonson Repr. Co., New York, 1979. pp. 295-354.
10. S. Ikeda, *Acta Phys. Polonica* B13 (1982), 321.
11. A. Kawaguchi, *Akad. Wetensch. Amsterdam Proc.* 40 (1937), 596.