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EFFECT OF HALL CURRENTS ON UNSTEADY HYDROMAGNETIC FLOW IN A ROTATING CHANNEL WITH OSCILLATING PRESSURE GRADIENT

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The effect of Hall currents on unsteady hydromagnetic flow of a viscous incompressible electrically conducting fluid in a rotating channel under the influence of a periodic pressure gradient and of a uniform magnetic field acting parallel to the axis of rotation is studied. The solution in dimensionless form contains four parameters viz. M (the Hartmann number), K^2 (the rotation parameter), ω (the frequency parameter) and m (the Hall current parameter). Asymptotic behaviour of the solution is analysed for large values of M^2, K^2 and ω.

1. INTRODUCTION

The problem of hydromagnetic flow in a rotating channel in the presence of an applied uniform magnetic field finds its applications in induction flow meters, MHD generators or accelerators, constructions of turbines and other centrifugal machines. Seth and Jana[1] have studied the unsteady hydromagnetic flow in a rotating channel neglecting Hall currents. However, in the presence of strong magnetic field, the Hall currents play a significant role in determining the flow features.

In the present paper we consider the effect of Hall currents on the unsteady hydromagnetic flow of an electrically conducting viscous incompressible fluid between two parallel infinite plates rotating with a uniform angular velocity Ω about an axis perpendicular to their planes, under the action of a periodic pressure gradient and of a uniform magnetic field acting parallel to the axis of rotation. Exact solution of the governing equations for the fully developed flow is obtained in closed form. The solution in dimensionless form contains four parameters viz. M^2, the square of Hartmann number, K^2, the rotation parameter, ω, the frequency parameter and m, the Hall parameter. The asymptotic behaviour of the solution is analysed for large values of M^2, K^2 and ω. It is found that there arise thin double boundary layers for large values of K^2 and ω near the plates of the channel. For large values of M^2, there

arise thin boundary layers of thickness $O\{M/\sqrt{1+m^2}\}^{-1}$ near the plates of the channel.

2. Formulation of the Problem and Its Solutions

Consider the unsteady fully developed flow of an electrically conducting viscous incompressible fluid within a parallel plate channel $z = \pm L$ due to a periodic pressure gradient along the x-direction when the fluid and channel rotate in unison about z-axis with a uniform angular velocity Ω. A uniform magnetic flux density B_0 is acting parallel to the axis of rotation. Since the plates are infinite along x and y directions, all physical variables except pressure depend on z and t only. Also the plates of the channel are electrically non-conducting. The following assumptions may be made which are compatible with the fundamental equations of magnetohydrodynamics :

$$\mathbf{q} = (u_x, u_y, 0), \quad \mathbf{B} = (B_x, B_y, B_0), \qquad \text{...(2.1)}$$

$$\mathbf{E} = (E_x, E_y, E_z), \quad \mathbf{J} = (J_x, J_y, 0)$$

where $\mathbf{q}$, $\mathbf{B}$, $\mathbf{E}$ and $\mathbf{J}$ are respectively, velocity of the fluid, magnetic induction vector, electric field and current density.

Taking magnetic Reynolds number to be very small, the induced magnetic field produced by the motion of electrically conducting fluid is neglected in comparison with the applied magnetic field so that we can take $\mathbf{B} = (0, 0, B_0)$. Since no external electric field is applied, the effect of polarization of fluid is neglected so that we can assume $\mathbf{E} = 0$ [see Meyer[2]].

Under the above assumptions, the equations of motion in dimensionless form, with the help of generalized Ohm's law taking Hall currents into account, become

$$\frac{\partial u}{\partial T} - 2K^2 v = -\frac{\partial p^*}{\partial \xi^*} + \frac{\partial^2 u}{\partial \eta^2} + \frac{M^2}{1+m^2}(mv - u) \qquad \text{...(2.2)}$$

$$\frac{\partial v}{\partial T} + 2K^2 u = \frac{\partial^2 v}{\partial \eta^2} - \frac{M^2}{1+m^2}(v + mu) \qquad \text{...(2.3)}$$

where $\xi^* = x/L$, $\eta = z/L$, $u = u_x L/\nu$, $v = u_y L/\nu$, $T = \nu t/L^2$, $p^* = L^2 p/\rho \nu^2$, $M = B_0 L (\sigma/\rho \nu)^{1/2}$ is the Hartmann number, $K^2 = \Omega L^2/\nu$ is the rotation parameter which is the reciprocal of Ekman number and $m = \omega_e \tau_e$ is the Hall parameter. Here ρ, ν, σ, t, ω_e, τ_e and p are respectively, fluid density, kinematic co-efficient of viscosity, electrical conductivity, time, cyclotron frequency, electron collison time and modified pressure including centrifugal force.

The boundary conditions are

$$u = v = 0 \quad \text{at } \eta = \pm 1. \qquad \text{...(2.4)}$$

Considering the pressure gradient in dimensionless form as

$$-\frac{\partial p^*}{\partial \xi^*} = R\, e^{i\omega T} \qquad \ldots(2.5)$$

we assume the velocity components in the following form

$$u(\eta, T) = u_1(\eta)\, e^{i\omega T} \text{ and } v(\eta, T) = v_1(\eta)\, e^{i\omega T} \qquad \ldots(2.6)$$

where $\omega = \omega' L^2/\nu$ is the frequency parameter and R is a constant.

Making use of eqns. (2.5) and (2.6), eqns. (2.2) and (2.3) reduce to

$$i\omega u_1 - 2K^2 v_1 = R + \frac{d^2 u_1}{d\eta^2} + \frac{M^2}{1+m^2}(m v_1 - u_1) \qquad \ldots(2.7)$$

$$i\omega v_1 + 2K^2 u_1 = \frac{d^2 v_1}{d\eta^2} - \frac{M^2}{1+m^2}(v_1 + m u_1). \qquad \ldots(2.8)$$

The boundary conditions (2.4) become

$$u_1 = v_1 = 0 \text{ at } \eta = \pm 1. \qquad \ldots(2.9)$$

Equations (2.7) and (2.8) together with the boundary conditions (2.9) are solved to give

$$u = \frac{R}{2}\left[\frac{(\alpha - i\beta)^2}{(\alpha^2+\beta^2)^2}\left\{1 - \frac{\cosh(\alpha + i\beta)\eta}{\cosh(\alpha + i\beta)}\right\} + \frac{(\alpha_1 \mp i\beta_1)^2}{(\alpha_1^2+\beta_1^2)^2}\left\{1 - \frac{\cosh(\alpha_1 \pm i\beta_1)\eta}{\cosh(\alpha_1 \pm i\beta_1)}\right\}\right] e^{i\omega T} \qquad \ldots(2.10)$$

$$v = \frac{iR}{2}\left[-\frac{(\alpha - i\beta)^2}{(\alpha^2+\beta^2)^2}\left\{1 - \frac{\cosh(\alpha + i\beta)\eta}{\cosh(\alpha + i\beta)}\right\} + \frac{(\alpha_1 \mp i\beta_1)^2}{(\alpha_1^2+\beta_1^2)}\left\{1 - \frac{\cosh(\alpha_1 \pm i\beta_1)\eta}{\cosh(\alpha_1 \pm i\beta_1)}\right\}\right] e^{i\omega T} \qquad \ldots(2.11)$$

where

$$\alpha, \beta = 2^{-1/2}\left[\left\{\frac{M^4}{1+m^2} + (\omega + 2K^2)\left(\frac{2mM^2}{1+m^2} + (\omega + 2K^2)\right)\right\}^{1/2} \pm \frac{M^2}{1+m^2}\right]^{1/2} \qquad \ldots(2.12a)$$

$$\alpha_1, \beta_1 = 2^{-1/2}\left[\left\{\frac{M^4}{1+m^2} - (\omega - 2K^2)\left(\frac{2mM^2}{1+m^2}\right.\right.\right.$$

(equation continued on p. 822)

$$- (\omega - 2K^2) \Big)\Big\}^{1/2} \pm \frac{M^2}{1 + m^2}\Big]^{1/2}. \qquad \ldots(2.12b)$$

The upper and lower signs in eqns. (2.10) and (2.11) are chosen as $\omega \gtrless 2K^2$ while the upper and lower signs in eqns. (2.12a) and (2.12b) are taken for α, α_1 and β, β_1 respectively.

Asymptotic Solutions

We shall now discuss a few particular cases of interest.

Case I : $K^2 \gg 1$, $M^2 \sim O(1)$, $\omega \sim O(1)$ and $m^2 \sim O(1)$.

When K^2 is large, M^2, ω and m^2 are of small orders of magnitude, the boundary layer type flow is expected. For the boundary layer on the plate $\eta = 1$, writing $(1 - \eta) = \xi$ we obtain

$$u = \frac{R}{4K^2}\left[- e^{-\alpha_1^* \xi} \sin\left(\omega T - \beta_1^* \xi\right) + e^{-\alpha_2^* \xi} \sin\left(\omega T + \beta_2^* \xi\right)\right] \qquad \ldots(2.13)$$

$$v = \frac{R}{4K^2}\left[-2\cos(\omega T + \pi) + e^{-\alpha_1^* \xi} \cos\left(\omega T - \beta_1^* \xi\right) + e^{-\alpha_2^* \xi} \cos(\omega T + \beta^* \xi)\right] \qquad \ldots(2.14)$$

where

$$\left.\begin{aligned} \alpha_1^* &= K\left(1 + \frac{1+m}{1+m^2}\,\frac{M^2}{4K^2} + \frac{\omega}{4K^2}\right) \\ \beta_1^* &= K\left(1 - \frac{1-m}{1+m^2}\,\frac{M^2}{4K^2} + \frac{\omega}{4K^2}\right), \\ \alpha_2^* &= K\left(1 + \frac{1+m}{1+m^2}\,\frac{M^2}{4K^2} - \frac{\omega}{4K^2}\right), \\ \beta_2^* &= K\left(1 - \frac{1-m}{1+m^2}\,\frac{M^2}{4K^2} - \frac{\omega}{4K^2}\right). \end{aligned}\right\} \qquad \ldots(2.15)$$

The expressions (2.13) and (2.14) show that the flow is divided into two regions viz. boundary layer region and central core region. The boundary layer region is confined to double boundary layers of thicknesses $O\left(\alpha_1^*\right)^{-1}$ and $O\left(\alpha_2^*\right)^{-1}$ which arise near

the plates of the channel. These boundary layers may be identified as modified hydromagnetic Ekman layers.

In the central core, given by $\xi \geqslant 1/\alpha_2^*$, the exponential terms in the expressions (2.13) and (2.14) damp out quickly and we obtain

$$u \simeq 0, \; v \simeq -\frac{R}{2K^2} \cos(\omega T + \pi). \qquad \ldots(2.16)$$

It is evident from the expression (2.16) that in the central core, the velocity in the direction of the pressure gradient vanishes away while it persists in the direction normal to the pressure gradient and the axis of rotation and it oscillates with the same frequency ω as the applied pressure gradient but has a phase lead of π over it. Thus in the central core, the fluid will be moving in a direction normal to the axis of rotation and applied pressure gradient.

Case 2 : $M^2 \gg 1$, $K^2 \sim O(1)$, $\omega \sim O(1)$ and $m^2 \sim O(1)$

In this case also the flow is of boundary layer type and we obtain

$$u = \frac{R}{2M^2}\Bigg[(2 - \omega^2 T^2)\left\{1 - \exp\left(-\frac{M\xi}{\sqrt{1+m^2}}\right)\left(1 - \frac{m^2 M^2 \xi}{8(1+m^2)}\right)\right\}$$

$$- \frac{\sqrt{1+m^2}}{2M}\left\{\left(\frac{m M^2}{1+m^2} + (\omega + 2K^2)\right) \pm \left(\frac{mM^2}{1+m^2} - (\omega - 2K^2)\right)\right\}$$

$$\omega T \xi \exp\left(-\frac{M\xi}{\sqrt{1+m^2}}\right)\Bigg] \qquad \ldots(2.17)$$

$$v = \frac{R}{4M^3}\sqrt{1+m^2}\left[\left(1 - \frac{\omega^2 T^2}{2}\right)\left\{\left(\frac{mM^2}{1+m^2} + (\omega + 2K^2)\right)\right.\right.$$

$$\left.\left. \mp \left(\frac{mM^2}{1+M^2} - (\omega - 2K^2)\right)\right\} \xi \exp\left(-\frac{M\xi}{\sqrt{1+m^2}}\right)\right]. \qquad \ldots(2.18)$$

The expressions (2.17) and (2.18) demonstrate the existence of a boundary layer of thickness of $O\{(M/\sqrt{1+m^2})\}^{-1}$ near the plate of the channel. This layer may be identified as modified Hartmann layer. It may be noted that the thickness of the boundary layer increases with in the increase in Hall parameter m while it decreases with the increase in Hartmann number M. It is interesting to note that the velocity in y-direction is very weak being $O(R/M^3)$. In the central core, given by $\xi > 1/(M/\sqrt{1+m^2})$, the velocity in y-direction vanishes away while it persists in the direction of applied pressure gradient and oscillates with the same frequency ω as the

applied pressure gradient. Thus in the central core, the fluid will be moving in the direction of pressure gradient only.

Case 3 : $\omega \gg 1$, $M^2 \sim O(1)$, $K^2 \sim O(1)$ and $m^2 \sim O(1)$

This case also has a boundary layer character and we obtain the velocity field as

$$u = \frac{R}{2\omega}\left[2\cos(\omega T - \pi/2) - e^{-\alpha_3\xi}\sin(\omega T - \beta_3\xi)\right.$$

$$\left. - e^{-\alpha_4\xi}\sin(\omega T - \beta_4\xi)\right] \qquad \ldots(2.19)$$

$$v = \frac{R}{2\omega}\left[e^{-\alpha_3\xi}\cos(\omega T - \beta_3\xi) - e^{-\alpha_4\xi}\cos(\omega T - \beta_4\xi)\right] \qquad \ldots(2.20)$$

where

$$\alpha_3 = \left(\frac{\omega}{2}\right)^{1/2}\left(1 + \frac{1+m}{1+m^2}\frac{M^2}{2\omega} + \frac{K^2}{\omega}\right)$$

$$\beta_3 = \left(\frac{\omega}{2}\right)^{1/2}\left(1 - \frac{1-m}{1+m^2}\frac{M^2}{2\omega} + \frac{K^2}{\omega}\right) \qquad \ldots(2.21)$$

$$\alpha_4 = \left(\frac{\omega}{2}\right)^{1/2}\left(1 + \frac{1-m}{1+m^2}\frac{M^2}{2\omega} - \frac{K^2}{\omega}\right)$$

$$\beta_4 = \left(\frac{\omega}{2}\right)^{1/2}\left(1 - \frac{1+m}{1+m^2}\frac{M^2}{2\omega} - \frac{K^2}{\omega}\right).$$

The expressions (2.19) and (2.20) reveal that the flow is divided into two regions viz. boundary layer region and central core region. The boundary layer region is confined to double boundary layers of thicknesses $O(\alpha_3)^{-1}$ and $O(\alpha_4)^{-1}$ which arise near the plate of the channel. These boundary layers may be identified as modified hydromagnetic Stokes-Ekman layers. The thickness of these layers decreases with the increase in M^2. The exponential terms in the expressions (2.19) and (2.20) damp out quickly as ξ increases when $\xi \geqslant 1/\alpha_4$ i. e. in central core, we obtain

$$u \approx \frac{R}{\omega}\cos(\omega T - \pi/2), \quad v \approx 0. \qquad \ldots(2.22)$$

It is evident from the expression (2.22) that in the central core region, the fluid has a velocity in primary flow direction which oscillates with the same frequency ω as the applied pressure gradient but has a phase lag of $\pi/2$ over it. Thus in the central region, the fluid will be moving in the direction of pressure gradient only.

3. Discussion of Results

To study the effects of Hall currents and oscillation on a hydromagnetic flow in a rotating channel, the non-dimensional velocity profiles are presented versus η in Figs. 1 and 2 for various values of m^2 and ω taking M^2, K^2 and ωT fixed. It is evident from Figs. 1 and 2 that the profiles of the velocity components u and v are flattened in the central region having maxima near the walls of the channel which indicate that there arise thin boundary layers near the walls of the channel. It is observed from Fig. 1 that for fixed value of ω, the primary velocity u is of oscillatory character with respect to m^2 while the secondary velocity v increases with the increase in m^2 whereas the velocity components u and v increase with the increase in ω for fixed values of m^2. Fig. 2 reveals that the primary velocity u decreases with the increase in either m^2

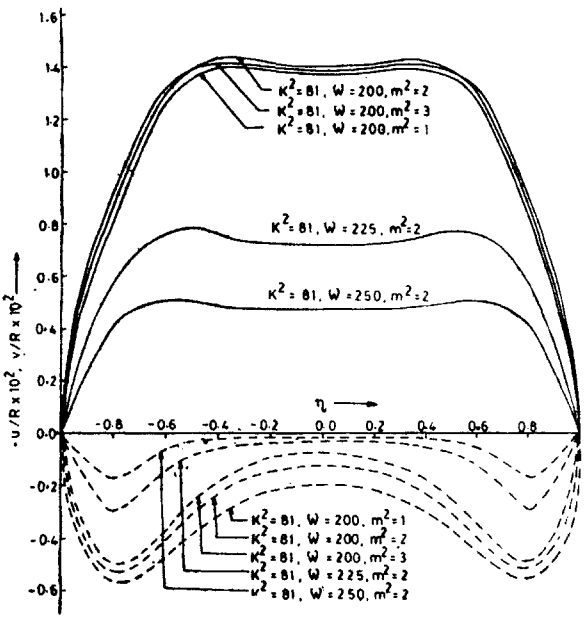

FIG. 1. Velocity Profiles ($-10^2u/R$ ..$10^2v/R$) when $M^2=10$, $\omega T=\pi/2$ and $\omega>2K^2$ for large K^2 and ω.

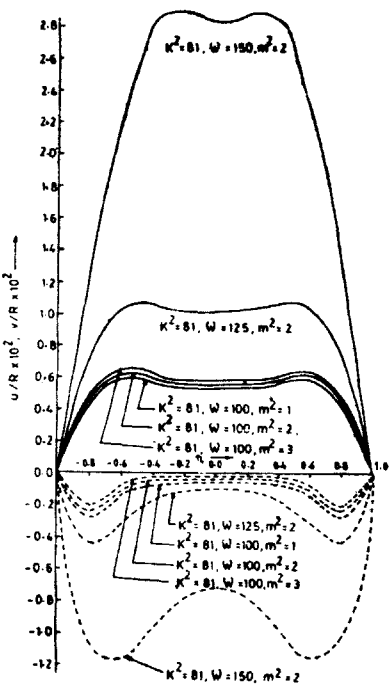

FIG. 2. Velocity profiles (-10^2 u/R,...$10^2v/R$) when $M^2=10$, $\omega T=\pi/2$ and $\omega<2K^2$ for large K^2 and ω.

or ω while for fixed values of ω, the secondary velocity v increases with the increase in m^2 and decreases with the increase in ω for fixed values of m^2.

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REFERENCES

1. G. S. Seth, and R. N. Jana, *Acta Mech.* **37** (1980), 29.
2. R. C. Meyer, *J. Aero/Space Sci.* (1958), 561.