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A NOTE ON THE PROPERTY OF AN ADJOINT OF THE ADJACENCY MATRIX OF STRONGLY REGULAR GRAPHS

N. S. CHAUDHARI, N. L. SARDA AND D. B. PHATAK

Department of Computer Science and Engineering
Indian Institute of Technology Bombay
Powai, Bombay 400076

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It is proved that exactly three distinct values are possible in the adjoint of an adjacency matrix of strongly regular graph. Their placement is given by the placement of zeros and ones in the corresponding adjacency matrix.

1. INTRODUCTION

Strongly regular graphs arise naturally in the investigations of many graph-theoretic problems. They are recognized to be hard cases for many heuristics proposed for the efficient solution of graph isomorphism problem[1,2,4].

In this note, we investigate the nature of distinct values in the adjoint of an adjacency matrix of strongly regular graphs.

2. NOTATIONS AND PRELIMINARIES

Let $G = (V, E)$ be an n-vertex simple graph in which no self-loops and multiple edges are permitted. Let $\mathbf{A}$ be a binary square matrix of size n, which is an adjacency matrix of G.

A vertex w in V is a neighbour of a vertex pair (u, v) iff (u, w) and (w, v) are edges in G.

A graph $G = (V, E)$ is said to be strongly regular (or, 2-strongly regular) with parameters (n, d, p, q) iff

(i) each vertex $v \in V$ has a degree d

(ii) each edge $(u, v) \in E$ has p distinct neighbours

(iii) each vertex pair $(u, v) \notin E$ has q distinct neighbours.

Let trace of a square matrix $\mathbf{M}$ of size n be denoted by tr $(\mathbf{M})$, where tr $(\mathbf{M}) = \sum_{i=1}^{n} \mathbf{M}[i, i]$.

A necessary and sufficient condition for an adjacency matrix $\mathbf{A}$ with zero diagonal entries to be an adjacency matrix of strongly regular graph is[3] :

$$\mathbf{A}^2 + (q - p)\,\mathbf{A} + (q - d)\,\mathbf{I} = q\,\mathbf{J} \qquad \ldots(2.1)$$

where **I** is an identity matrix of size n and **J** is a square matrix of size n which contains all ones.

3. A Theorem

Let **A** be an adjacency matrix of a strongly regular graph $G = (V, E)$ with parameters (n, d, p, q). Let **B** = adjoint (**A**), where the arithmetic operations for computing the adjoint are defined over the field of rationals. Then **B** contains exactly three distinct rational numbers, say a, b, and c whose placement in the matrix **B** is given below :

(I) $\mathbf{B}[i, j] = a$ for all i, j, such that $\mathbf{A}(i, j] = 1,\ i \neq j$.

(II) $\mathbf{B}[i,j] = b$ for all i, j, such that $\mathbf{A}[i, j] = 0,\ i \neq j$.

(III) $\mathbf{B}[i, j] = c$ for all i, j, such that $i = j$.

Proof : Let **J** be a square matrix of size n which contains all ones. Then, since each vertex of G has degree d, we have :

$$\mathbf{J}\,\mathbf{A} = d\,\mathbf{J}. \qquad \ldots(3.1)$$

Hence,

$$\mathbf{J}\,\mathbf{A}^k = d^k\,\mathbf{J}. \qquad \ldots(3.2)$$

Next, let us prove the following result for all m :

[I] $\mathbf{A}^m[i, j] = x_m$ for all i, j, such that $\mathbf{A}[i, j] = 1,\ i \neq j$ .. (3.3.1)

(II) $\mathbf{A}^m[i, j] = y_m$ for all i, j, such that $\mathbf{A}[i, j] = 0,\ i \neq j$ ...(3.3.2)

(III) $\mathbf{A}^m[i, j] = z_m$ for all i, j, such that $i = j$ ...(3.3.3)

The above result is trivially true for $m = 1$, where $x_1 = 1$, $y_1 = 0$, $z_1 = 0$.

For $m = 2$, let us examine the entries in $\mathbf{A}^2$.

$$A^2 = q\,\mathbf{J} - (q - d)\,\mathbf{I} - (q - p)\,\mathbf{A} \qquad \ldots(3.4)$$

From the above equation, it is obvious that

(I) $\mathbf{A}^2[i, j] = p$ for all i, j, such that $\mathbf{A}[i, j] = 1,\ i \neq j$

(II) $\mathbf{A}^2[i,j] = q$ for all i, j,

such that $\mathbf{A}[i, j] = 0$, $i \neq$

(III) $\mathbf{A}^2[i, j] = d$ for all i, j.

such that $i = j$.

Thus, the result in (3.3) is true for $m = 2$ with $x_2 = p$, $y_2 = q$, $z_2 = d$.

Next, let us assume that (3.3) is true for all $m \leqq k + 1$.

From (3.4), we have,

$$\mathbf{A}^{k+2} = q\,\mathbf{J}\mathbf{A}^k - (q - d)\,\mathbf{A}^k - (q - p)\,\mathbf{A}^{k+1}$$

$$= qd^k\,\mathbf{J} - (q - d)\,\mathbf{A}^k - (q - p)\,\mathbf{A}^{k+1}. \qquad \ldots(3.5)$$

From (3.5), it follows that,

(I) $x_{k+2} = qd^k - (q - d)\,x_k - (q - p)\,x_{k+1}$

(II) $y_{k+2} = qd^k - (q - d)\,y_k - (q - p)\,y_{k+1}$

(III) $z_{k+2} = qd^k - (q - d)\,z_k - (q - p)\,z_{k+1}$.

Hence, (3.3) follows from induction

Next, from Cayley-Hamilton theorem[5] the adjoint (A) can be written as; adjoint

$$(\mathbf{A}) = \mathbf{B} = (-1)^{n-1}(\mathbf{A}^{n-1} + w_1(\mathbf{A})^{n-2} \ldots + w_{n-2}\,\mathbf{A} + w_{n-1}\,\mathbf{I}) \qquad \ldots(3.6)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are scalars. In particular they can be computed as follows[3]

$$\mathbf{Q}_1 = \mathbf{I}$$

$$\alpha_1 = -\operatorname{tr}(\mathbf{Q}_1\,\mathbf{A})$$

$$\mathbf{Q}_2 = \mathbf{Q}_1\,\mathbf{A} + \alpha_1\,\mathbf{I}$$

$$\alpha_2 = -\tfrac{1}{2}\operatorname{tr}(\mathbf{Q}_2\,\mathbf{A})$$

$$\vdots$$

$$\mathbf{Q}_{n-1} = \mathbf{Q}_{n-2}\,\mathbf{A} + \alpha_{n-2}\,\mathbf{I}$$

$$\alpha_{n-1} = -\frac{1}{n-1}\operatorname{tr}(\mathbf{Q}_{n-1}\,\mathbf{A})$$

$$\mathbf{Q}_n = \mathbf{Q}_{n-1}\,\mathbf{A} + \alpha_{n-1}\,\mathbf{I}$$

and, we have adjoint $(\mathbf{A}) = \mathbf{B} = (-1)^{n-1}\,\mathbf{Q}_n$.

From (3.6) and (3.3), we have

(I) $\mathbf{B}[i, j] = (-1)^{n-1}(x_{n-1} + w_1\,x_{n-2} + \ldots + w_{n-2}\,x_1)$

for all i, j,

such that $\mathbf{A}[i, j] = 1, i \neq j$

$= a$, say, for all i, j,

such that $\mathbf{A}[i, j] = 1, i \neq j$

(II) $\mathbf{B}[i, j] = (-1)^{n-1}(y_{n-1} + w_1 y_{n-2} + \ldots + w_{n-2} y_1)$

for all i, j,

such that $\mathbf{A}[i, j] = 0, i \neq j$

$= b$, say, for all i, j,

such that $\mathbf{A}[i, j] - 0, i \neq j$

(III) $\mathbf{B}[i, j] = (-1)^{n-1}(w_1 z_{n-1} + \alpha_1 z_{n-2} + \ldots + w_{n-2} z_1 + w_{n-1})$

for all i, j

such that $i = j$

$= c$, say, for all i, j,

such that $i = j$.

Hence the theorem.

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