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AN EXTENSION OF KATAI-IWAI PROCEDURE TO DERIVE BALANCING AND MINIMUM BALANCING SETS OF A SOCIAL SYSTEM*

B. DEVADAS ACHARYA†

The Mehta Research Institute of Mathematics and Mathematical Physics
26, *Dilkusha, New Katra, Allahabad* 211002

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Katai and Iwai[7,8] discovered a method to derive balancing sets and minimum balancing sets of a sociogram with planar graph structure. In this paper, we offer a new perspective to apply their method to sociograms with arbitrary graph structures. For this purpose, we use the notion of 'marked intersection multigraph' of a cycle basis of the underlying graph and the notion of p-balance of a marked graph due to Sampathkumar[16].

For all terminology and notation in graph theory, we follow Harary[6].

A graph $G = (V, E)$ together with a function $f: V \cup E \to \{-1, 1\}$ is called a *net*, denoted G_f, where f is called a sign function of G. For an *element* $x \in V \cup E$, $f(x)$ is called the *value* or the *sign* of x in G_f and accordingly x is said to be *positive* or *negative* in G_f depending on whether $f(x) = 1$ or -1, respectively. Further, the *sign of a subgraph* H in G_f, denoted $f(H)$, is defined to be the product of the signs of the elements in H. A net G_f in which all the lines (points) are positive is called a *marked (signed)) graph* and f is then called a *marking (signing)* of G. We call G_f a *net on* G and G is called the *underlying graph* of G_f. Given a graph G, $\Phi(G)$ (respectively, $\Psi(G)$) will denote the set of all marked (singed) graphs on G.

A marked (signed) graph G_f is said to be *p-balanced* (*balanced*) if every component (cycle) of G_f has an even number of negative points (lines). Balanced signed graphs were first characterized by Harary[4]

Theorem 1 (Harary[4])—A signed graph G_f is balanced if and only if its point set $V(G)$ can be partitioned into two subsets V_1 and V_2 (one of them possibly empty) such that every positive line has its ends in V_i, $i = 1$ or 2, and every negative line has one end in V_1 and the other in V_2.

(See Harary *et al.*[5], Peay[11,12], Katai and Iwai[7,8], Acharya[1], Zaslavsky[14,15], McKee[9,10] for various other characterizations of balanced signed graphs.)

Theorem 2 (Sampathkumar[16])—A marked graph G_f is p-balanced if and only if there is a signing g of G such that

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†*Present Address* : Department of Science & Technology, New Mehrauli Road, New Delhi-110016.

$$f(u) = \prod_{v \in N(u)} g(u\,v) \text{ for all } u \in V(G)$$

where $N(u) = \{w \in V(G) | uw \in E(G)\}$ is the open neighbourhood of u.

The notion of balance in a signed graph is, in fact, a mathematical model of a state of psychological consonance in a social group resulting from a dichotomous interpersonal interaction pattern among its individuals[3,7,8]. Thus, when the social group is not in such a state of consonance, there is psychological dissonance in the group which creates a tendency among the individuals to readjust their interrelations with others in the group so as to settle in a state of consonance. Such processes are known as *balancing processes* which may be identified with sets of interrelated pairs of individuals whose interpersonal attitude changes could bring about a state of psychological state of consonance amomg the individuals in the social group. Mathematically, this amounts to locating the sets of lines in the signed graph, representing the structure of interpersonal interactions within the social group, whose signs when changed would yield a balanced signed graph—such sets of lines in a signed graph in general are called *balancing sets*. For example, in the unbalanced signed graph S shown in Fig. 1 (a), the set $\{ae, df\}$ is a balancing set, where positive and negative lines of S are distinguished by drawing them as solid and broken line segments, respectively. (This is the standard convention.) The balanced signed graph S' obtained after changing the signs of the lines in this balancing set is shown in Fig. 1 (b).

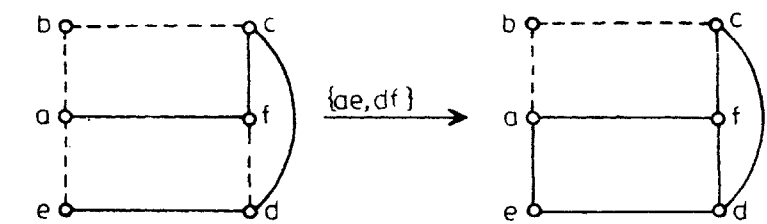

FIG. 1.

When the underlying graph of a signed graph G_f is planar, a simple procedure to derive a balancing set was given by Katai and Iwai[7,8] and consists in performing the following steps :

Step I: Take any plane imbedding of G_f, let $Z_1, Z_2, \ldots, Z_k$ be the cycles whose interiors on the plane are the faces of G_f in that imbedding and let Z_0 denote the exterior face.

Step II : Draw the 'marked dual' $D_F(G_f)$ of G_f, corresponding to the plane imbedding of G_f mentioned in Step I, with $\{Z_0, Z_1, Z_2, \ldots, Z_k\}$ as its point set, with Z_i

joined to Z_j by a line labeled x if x is a line of G common between Z_i and Z_j for any two distinct indices $i, j \in \{0, 1, 2, \ldots, k\}$, and with the marking F defined by assigning the sign $f(Z_i)$ of Z_i to Z_i for each $i \in \mathbf{k} = \{1, 2, \ldots, k\}$ and the product $\prod_{i=1}^{k} f(Z_i)$ to Z_0.

Lemma 1 (Katai and Iwai[7,8])—If G is connected, then $D_F(G_f)$ has an even number of negative vertices which may be partitioned into two-element subsets $\{Z_{i_1}, Z_{j_1}\}$, $\{Z_{i_2}, Z_{j_2}\}, \ldots, \{Z_{i_r}, Z_{j_r}\}$ such that (i) there is a $Z_{j_m} Z_{i_m}$ —path P_m in $D_F(G_f)$ for each $m \in \mathbf{r}$ and (ii) these paths $P_1, P_2, \ldots P_r$ are pairwise line disjoint.

This lemma enables to perform the next step, viz.

Step III : Partition the set of negative points of $D_F(G_f)$ into two-element subsets $\{Z_{i_1}, Z_{j_1}\}, \{Z_{i_2}, Z_{j_2}\}, \ldots, \{Z_{i_r}, Z_{j_r}\}$ and find a set of pairwise line disjoint paths $P_1, P_2, \ldots P_r$ such that P_m joins Z_{i_m} and Z_{j_m} for all $m \in \mathbf{r}$.

Step IV : List the lines of G_f which correspond to the lines of $\bigcup_{m=1}^{r} E(P_m)$ in $D_F(G_f)$.

Theorem 3 (Katai and Iwai[7,8])—For a planar graph G, and for any signing f of G. the output of the algorithm is a balancing set of G_f.

For example, consider the 'planar signed graph' G_f of Fig. 1 (a), imbedded on the plane as shown there. We have the cycles $Z_1 = (a, b, c, f, a)$, $Z_2 = (c, d, f, c)$, $Z_3 = (a, e, d, f, a)$ representing the finite faces of the imbedded signed graph and Z_0, the 'exterior face' of the figure, may be imagined to be bounded by the outer cycle (a, b, c, d, e, a) which, in this case, is the (vector) sum (as computed in the Galois field $GF(2)$) of the line sets of the cycles Z_1, Z_2 and Z_3, written as $Z_0 = Z_1 + Z_2 + Z_3$ or as $\sum_{i=1}^{3}{}^{*} Z_i$. Then the marked dual of G_f is shown in Figure 2. The Z_0 Z_2-path Z_0 $\overrightarrow{ae}$ Z_3 $\overrightarrow{df}$ Z_2 consisting of the lines labeled ae and df is the one that gives us the balancing set $\{\overrightarrow{ae}, \overrightarrow{df}\}$ of G_f. A shorter Z_0 Z_2-path is one containing the single line labeled cd, and accordingly this gives a 'minimum balancing set' $\{cd\}$ of G_f (thus, verify that the change of the sign of cd in G_f yields a balanced signed graph). For more complicated examples, we refer the interested reader to Katai and Iwai[8].

Observe that the success of the above procedure rests on Lemma 1 which is a property of marked duals of a planar signed graph. Hence, we can hope to extend the procedure to arbitrary signed graphs provided we extend the notion of marked duals of planar signed graphs. We offer a way here towards this end.

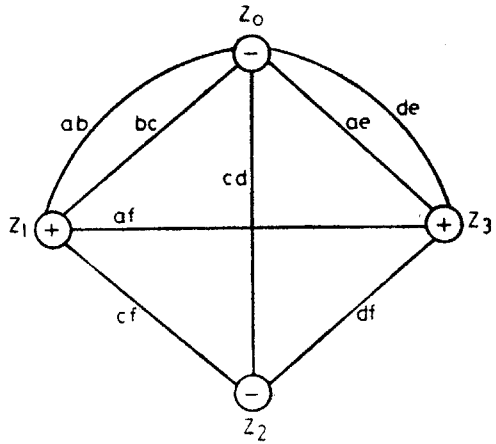

FIG. 2. The marked dual of G_f of Figure 1 (a).

With every graph G, we can associate a vector space $\theta(G)$ of all cycles and unions of line disjoint cycles of G, called the cycle space of G. $\theta(G)$ is generated by the set of all cycles of G and the cycles of G are the minimal elements of $\theta(G)$ in the sense that no cycle properly contains a nonzero element of $\theta(G)$. A 'cycle basis' of G is defined as a basis for $\theta(G)$ which consists entirely of cycles.

If a graph G is not 2-connected, i. e. if G is not connected and/or G contains a cutpoint, then a cycle basis π of G can be partitioned into bases $\pi_1, \pi_2, \ldots, \pi_p$ of cycle subspaces $\theta_1, \theta_2, \ldots, \theta_p$ generated by the cycles in 2-connected components (i. e. blocks) $B_1, B_2, \ldots, B_p$ of G. Therefore, $\theta(G)$ is the direct sum of cycle spaces of the blocks of G. Since the notion of balance in a signed graph G_f on G is defined in terms of the signs of the cycles in G_f, this decomposition implies that the generality is not alleviated if we assume 2-connectendness of G, and hence we shall assume hither-forth that G is a block.

If G is planar, then for a given embedding of G in the plane, the set of interior faces represented by their bounding minimal cycles gives rise to a cycle basis of G, called a 'planar cycle basis' of G. The following version of MacLanes's well known characterization of planar blocks characterizes also planar cycle bases of a graph.

Theorem 4 (cf. Syslo[13])—A block G is planar if and only if G contains a family of cycles $\pi^* = \{Z_0, Z_1, Z_2, \ldots, Z_k\}$ such that $\pi = \{Z_1, Z_2, \ldots, Z_k\}$ is a cycle basis and every line of G occurs in exactly two cycles of π^*.

We observe here that Z_0 appearing in the statement of Theorem 4 is, in fact, the vector sum of the basis cycles $Z_1, Z_2, \ldots, Z_k$ in the given embedding of G. This is the point from where we take off as follows : Let G be any graph (with 2-connectedness assumed) and π be any cycle basis of G. Define $Z_0 := Z_0(\pi) = \sum_{Z \in \pi}^{*} Z$ and let $\pi^* = \pi \cup \{Z_0\}$. Now, given $G_f \in \Psi(G)$ construct the 'marked intersection multigraph'

$\pi^*(G_f)$ *of* G_f precisely in the manner in which $D_F(G_f)$ was constructed in STEP II above when G_f was planar. In view of Theorem 4, it follows that the label of a line of G_f may appear in general on more than one line of $\pi^*(G_f)$. Of course, every line of G_f occurs in at least two members of π^* when G is nonplanar. We say that a line x of G_f is 'admissible' if it occurs in precisely two members of π^*, otherwise it is said to be 'inadmissible'. Clearly, a line x of G_f is inadmissible if and only if $\pi^*(G_f)$ contains at least two lines with label x. Accordingly, we shall refer to a line of $\pi^*(G_f)$ as 'reliable' if its label corresponds to an admissible line of G_f, otherwise the line is said to be 'unreliable'. Let $\underline{\pi}^*(G_f)$ denote the spanning subgraph of $\pi^*(G_f)$ consisting of the reliable lines of $\pi^*(G_f)$. A few important observations can be made now.

Lemma 2—Let G be a block, $G_f \in \Psi(G)$ and π be any cycle basis of G. Then $\pi^*(G_f)$ is p-balanced.

PROOF : Since G is a block it follows that $\pi^*(G_f)$ is connected and hence the assertion follows by the definition of the sign of Z_0 in $\pi^*(G_f)$.

Proposition 1—If G_f is a planar block then $\pi^*(G_f) = \underline{\pi}^*(G_f) = D_F(G_f)$ for any planar cycle basis π of G.

PROOF : This follows from Theorem 4.

We require the following result of Sampathkumar[16] for proving the main result of this paper.

Lemma 3 (Sampathkumar[16])—Any even number $2r$ of points in a connected graph G can be partitioned into two-element subsets $\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_r, v_r\}$ such that there is a $u_i v_i$-path P_i in G for each $i \in \mathbf{r}$, and for distinct $i, j \in \mathbf{r}$, P_i and P_j are line disjoint.

Theorem 5—Let G be any block, $G_f \in \Psi(G)$ and π be a cycle basis of G such that $\underline{\pi}^*(G_f)$ is p-balanced. Then there is a 'pairing' $\{\{Z_{i_1}, Z_{j_1}\}, \{Z_{i_2}, Z_{j_2}\}, \ldots, \{Z_{i_r}, Z_{j_r}\}\}$ of the negative points of $\underline{\pi}^*(G_f)$ and Z_{i_m} Z_{j_m}-paths P_m, $m \in \mathbf{r}$, which are pairwise line disjoint. Furthermore, the set of admissible lines of G_f that correspond to the (reliable) lines in $\bigcup_{m=1}^{r} E(P_m)$ constitute a balancing set of G_f.

PROOF : By hypothesis, $\underline{\pi}^*(G_f)$ is p-balanced and hence every component of $\underline{\pi}^*(G_f)$ has an even number of negative points. Therefore, by applying Lemma 3 to each component of $\underline{\pi}^*(G_f)$ we get a pairing $\{\{Z_{i_1}, Z_{j_1}\}, \{Z_{i_2}, Z_{j_2}\}, \ldots, \{Z_{i_2}, Z_{j_2}\}$ of the negative points of $\pi^*(G_f)$ and Z_{i_m} Z_{j_m} -paths P_m, $m \in \mathbf{r}$, which are pairwise line disjoint.

Now, for a given $m \in \mathbf{r}$, consider the path $P_m = \left(Z_{t_m} = Z_m^0, x_m^1, Z_m^1, x_m^2, Z_m^2, x_m^3, \ldots, x_m^{t(m)}, Z_m^{t(m)} = Z_{j_m} \right)$, $t(m)$ an integer depending on m, in $\pi^*(G_f)$ that joins the negative points Z_{t_m} and Z_{j_m}. Then, the changes of the sign of admissible line x_m^1 in G_f results in change of the sign of the cycle Z_{t_m} from negative to positive and also that of the cycle Z_m^1 from positive to negative or vice versa. Next, we change the sign of the admissible line x_m^2 in G_f. This changes the sign of the cycle Z^1 which at this stage is negative, from negative to positive (or vice versa respectively) and that of Z_m^2 from positive to negative vice versa. In this way, choosing successively the admissible lines $x_m^3, x_m^4, \ldots, x_m^{t(m)}$ in G_f and changing their sings we find that all the cycles $Z_m^2, Z_m^3, \ldots, Z_m^{t(m)-1}$ of G_f retain their original signs (given m, of course, at this stage) as each of them undergoes precisely two sign changes while each of the negative cycles $Z_m^0 = Z_{t_m}$ and $Z_m^{t(m)} = Z_{j_m}$ of P_m undergoes exactly one sign change so becoming positive. Thus, the change of signs of the admissible lines $x_m^1, x_m^2, \ldots, x_m^{t(m)}$ in G_f transforms the negative cycles Z_{t_m}, Z_{j_m} into positive ones. This argument holds for each $m \in \mathbf{r}$. Moreover, since π is a cycle basis of G and since the paths P_m, $m \in \mathbf{r}$, are all pairwise line disjoint with the pairs $\{Z_{t_m}, Z_{j_m}\}$ of their end points forming a partition of the set of negative vertices of $\pi^*(G_f)$, we see that $L = \bigcup_{m=1}^{r} \left\{ x_m^1, x_m^2, \ldots, x_m^{t(m)} \right\}$ is a balancing set of G_f.

A balancing set L of a signed graph G_f is said to be 'minimum' if it contains fewest possible lines among such sets in G_f. The problem of finding a minimum balancing set of G_f may be formulated as a shortest route problem in the general case too following exactly the method propounded by Katai and Iwai[8] for the case of planar signed graphs. We omit from here the details of the proposed method for a diligent reader to verify.

We now illustrate Theorem 5 by means of an example. Consider the signed graph G_f on $G = K_5$ the complete graph of order 5, which is the smallest nonplanar graph, shown in Fig. 3 (a). $\pi^*(G_f)$ and $\underline{\pi}^*(G_f)$ for a particular cycle basis π of G, are shown in Figure 3 (b and c) respectively. In this case, $\underline{\pi}^*(G_f)$ is connected and hence p-balanced. Moreover, for the pairing $\{\{Z_2, Z_5\}, \{Z_3, Z_6\}\}$ of the set of negative points of $\underline{\pi}^*(G_f)$, we have line disjoint paths $P_1 = (Z_2, x_2, Z_5)$ and $P_2 = (Z_3, x_7, Z_4, x_4, Z_6)$ connecting Z_2 to Z_5 and Z_3 to Z_6, respectively. It is then easily verified that the set

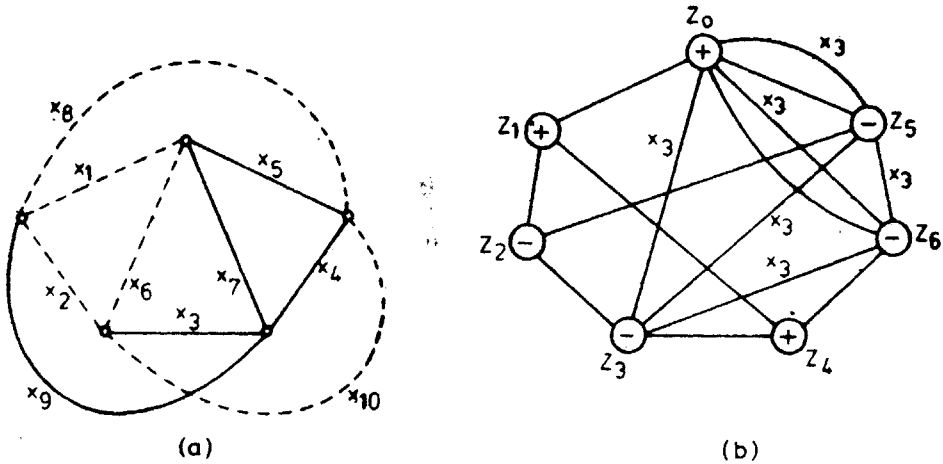

FIG. 3 (a) $\pi = \{Z_1 = (x_1, x_5, x_8), \quad Z_2 = (x_1, x_2, x_6)$
$Z_3 = (x_3, x_6, x_7), \quad Z_4 = (x_4, x_5, x_7),$
$Z_5 = (x_2, x_3, x_9), \quad Z_6 = (x_3, x_4, x_{10})\}$
$Z_0(\pi) := Z_0 = (x_8, x_9, x_3, x_{10})$

FIG.3(b) The marked intersection multigraph $\pi^*(G_f)$ of G_f in (a)

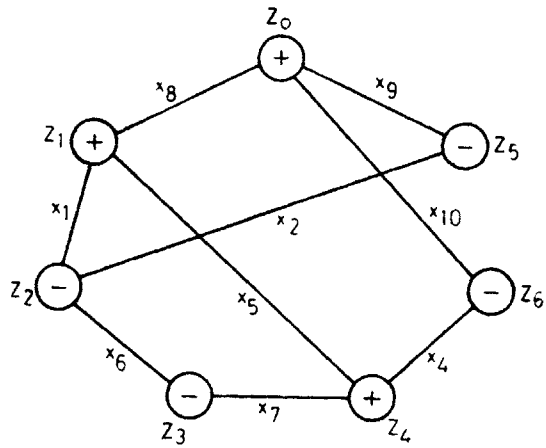

FIG. 3(c) The spanning subgraph $\underline{\pi}^*(G_f)$ of $\pi^*(G_f)$ consisting of only the reliable lines.

$L = \{x_2, x_4, x_7\}$ is a balancing set of G_f. Incidentally, it is also a minimum balancing set of G_f.

It appears that for any 2-connected signed graph there always exists a cycle basis satisfying the hypothesis of Theorem 5. If this conjecture turns out to be true the

method suggested in the proof of Theorem 5, for deriving balancing sets of signed graphs, would be universal. It would then be a valuable procedure, especially for 'socio-engineers', for easy implementation once such a cycle basis has been found. Katai-Iwai method for the general case, on the other hand, requires quite tedious analysis and calculations involving eigen values and eigen vectors of the adjacency matrix of the signed graph as also factor analytic techniques to derive its minimum balancing sets[8].

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