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TWO-PIECE PROPERTY IN MANIFOLDS WITHOUT FOCAL POINTS

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In this paper the concept of two-piece and spherical two-piece properties have been established and studied for submanifolds of Riemannian manifolds without focal points. Geodesic spheres, balls, horospheres and horodiscs in manifolds without focal points are proved to have spherical two-piece property. Totally geodesic hypersurfaces in hyperbolic space are proved to have the spherical two-piece property as well.

1. INTRODUCTION

Banchoff[1] started the subject of two-piece and spherical two-piece properties for submanifolds of the Euclidean and spherical spaces E^n and S^n, respectively. He defined the above mentioned properties in his paper[1] as follows :

(i) A set A in E^{n+1} (or S^{n+1}) has the spherical two-piece property (*STPP*) if $A \cap \bar{D}_i$ is pathwise connected for any complementary component D_i ($i = 1, 2$) of a hyperplane or hypersphere S in E^{n+1} (or S^{n+1}).

(ii) A set A is said to have the two-piece property (*TPP*) if $A \cap \bar{D}_i$ is pathwise connected for any complementary component D_i ($i = 1, 2$) of a hyperplane H^n of E^{n+1}.

Accordingly, every subset of a Euclidean space with spherical two-piece property has the two-piece property. The converse statement is not true generally as-for instance-the ellipsoide in E^3 has the two-piece property while it has not the spherical two-piece property. It is easy to find examples in Euclidean space showing that the union and intersection of subsets with *STPP* are not necessarily of *STPP*.

Adopting the above definitions, Banchoff[1] proved the following results :

Theorem 1.1—An n-sphere S^n in E^{n+1} has the *STPP* if $n > 0$. A hyperplane H^n in E^{n+1} has the *STPP*.

Theorem 1.2—A set A contained on an S^n in E^{n+1} has the *TPP* as a subset of E^{n+1} if and only if A has the *STPP* as a subset of S^n.

Theorem 1.3—If A is a smoothly embedded 2-sphere in E^n which has the *STPP*, then A must be Euclidean 2-sphere.

The study of the *STPP* and *TPP* has been developed during the last few years by other mathematicians[2-5]. The aim of this work is to extened and generalize the above mentioned study to include manifolds without focal points. In particular what we are

going to prove is true in hyperbolic spaces. This conclusion will become clear when we give the indicating figures at the end.

2. Preliminaries

In this section we quote briefly the necessary facts concerning manifolds without focal points as they represent the ambient space of all immersions considered in this paper. For more details we refer the reader to[2,6,7,8]. For the rest of this section, let W be a C^∞ complete simply connected Riemannian manifold without conjugate points unless we mention other properties.

Let $d(p, q)$ denote the distance between the two points $p, q \in W$. For each element v of the unit sphere bundle SW of W and for each real number $s > 0$, let us define the real-valued functions $b_{vs} : W \to \mathbf{R}$ by $b_{vs}(q) = s - d(\gamma_v(s), q)$, where γ_v is the maximal geodesic of W with initial velocity $\gamma_v'(0) = v$. The functions b_{vs} are increasing with s and absolutely bounded by $d(\gamma_v(0), q)$. The Busemann function of v is defined by $b_v = \lim_{s\to\infty} b_{vs}$. Each b_v is C^1 function defined on the whole of W. In particular if W is E^n, each b_v represents the usual height function in the direction of v. Call $H_v = b_v^{-1}(0)$ the horosphere and $B_v = b_v^{-1}(0, \infty)$ the horodisc of v. A vector $w \in S_qW$ for the point $q \in w$ is called asymptotic to v if $\nabla b_{vsi}(q) \to w$ for some sequence $s_i \to \infty$. Geometrically, this would mean that the horospheres passing through q determined by both v and w are coincident. A vector $w \in SW$ asymptotic to $v \in SW$ is called bi-asymptotic if also $-w$ is asymptotic to $-v$. Also we call the geodesics γ_v and γ_w asymptotic (or bi-asymptotic) if so are v and w. It is worth mentioning that Goto[7] proved the following :

Theorem 2.1—Let W be a complete, simply connected, Riemannian manifold without focal points. Given a geodesic ray γ and a point $p \in M$, there exists a unique geodesic ray asymptotic to γ passing through p.

Theorem 2.2—Let W be a complete simply connected Riemannian manifold without focal points. Then any two distinct geodesic rays starting from any $p \in W$ can not be asymptotic to each other.

In this paper, we require that the Busemann functions behave nicely. To ensure this requirement we assume that the manifold W-beside being without conjugate points-has bounded aspmptote (for details see Eschenburg[6]). The bounded asymptote condition is not to restrictive as manifolds without focal points as well as manifolds whose geodesic flow is Anosov all have the bounded asymptote property.

The condition that W has bounded asymptote is enough to ensure that :

(a) each Busemann function is C^2 and has gradient vector field of unit length; (b) the level hypersurfaces (horospheres) of each Busemann function form an equidistant family whose orthogonal trajectories are geodesics; (c) if u is a unit vector at $p \in W$,

then $u = \text{grad}\, b_u(p)$. Moreover, if $v = \text{grad}\, b_u(q)$ for some $q \in W$ then b_v differ only by a constant. This would mean that horospheres determined by b_u are the same as those determined by b_v.

One of the important facts we are going to use is the following.

Theorem 2.3—Let N, P be submanifolds of a Riemannian manifold M, and let us consider only broken C^∞ rectangles whose initial and final transversals are in N and P. Then a curve τ has the property that for all such rectangles with base curve τ, $l'(0) = 0$ if and only if τ is a geodesic from N to P which is perpendicular to both N and P. (In this theorem, $l(y)$ denotes the length of the longitudinal curve τ_y, $y \in [0,1]$ of the broken C^∞ rectangle $Q : [a, b] \times [0, 1] \to M$, (see Bishop and Crittenden[2]).

Now, we proceed to define the spherical two-piece and two-piece properties for submanifolds of a complete, simply connected Riemannian manifold W without focal points. If we look at the horosphere as being the limit of a geodesic sphere sequence, we can use horospheres in W to play the same part of hyperplanes in Euclidean space. Consequently, the two-piece-property in W (or h-two-piece property as being denominated by Bolton[3]) can be defined as follows :

(i) A submanifold M of W has the *TPP* if for every $v \in SW$ and $t \in \mathbf{R}$, $M \cap (W \setminus b_v^{-1}(t))$ has at most two components.

Similarly, we can define the spherical two-piece property as follows :

(ii) A submanifold M of W has the *STPP* if for every $v \in SW$, $t, r \in \mathbf{R}$, $r > 0$ and $x \in W$, each of the following sets $M \cap (W \setminus S_x(r))$ and $M \cap (W \setminus b_v^{-1}(t))$ has at most two components.

Bolton[3] has defined generalized Gauss map for immersed hypersurfaces into manifolds with bounded asymptote. Moreover, the relation between the above mentioned and the h-two-piece property has been established.

3. Main Work

The main part of this work is incorporated in establishing the following results:

Theorem 3.1—A geodesic sphere in a complete simply connected Riemannian manifold W without focal points has the *STPP*.

Proof : Let $S_r(x)$ be a geodesic sphere in W of radius r and center $x \in W$. In contrary to the theorem, assume that $S_r(x)$ has not the *STPP*. This assumption would mean that one (or both) of the following to possibilities will occur :

(i) there exists a geodesic sphere S' of W such that $S_r(x) \cap (W \setminus S')$ has more than two components;

(ii) there exists a horosphere H of W such that $S_r(x) \cap (W \setminus H)$ has more than two components.

To complete the proof we show that neither (i) nor (ii) could happen.

If (i) occurs, we should arrive at the situation represented in Fig. 1. Applying Theorem 2.3, there exists two geodesic rays γ_1 and γ_2 perpendicular to both of $S_r(x)$ and S. This would require that $S_r(x)$ and S' have the same center x. As $S_r(x) \cap S' \neq \phi$, we conclude that $S_r(x) \equiv S'$. Hence, $S_x(q) \cap (W \setminus S') = \phi$ which represents a contradiction and (i) is impossible.

If (ii) occurs, there will be a horosphere H such that $S_r(x) \cap (W \setminus H)$ has more than two components, i.e. we have a situation represented in Fig. 2. Using Theorem 2.3 again, there exist two goedesic rays γ_1 and γ_2 perpendicular to both $S_r(x)$ and H and meeting at x. This situation would mean that from the point x there exist two different asymptotic geodesic rays. This is a contradiction to Theorem 2.2 and the proof of Theorem 3.1 is complete.

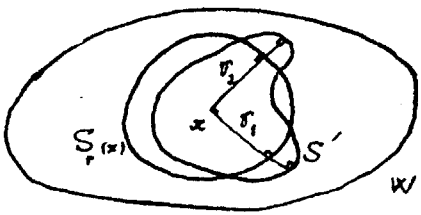

FIG. 1.

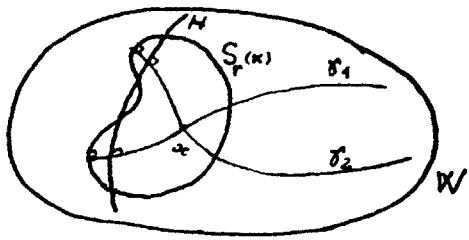

FIG. 2.

Corollary 3.1 – Closed geodesic balls in W have the *STPP*.

PROOF : Let $\bar{B}_r(x)$ be a closed geodesic ball in W. Assume in contrary that $\bar{B}_r(x)$ has not the *STPP*. This would mean that the boundary $\partial\bar{B}_r(x) = S_r(x)$ which is a geodesic sphere has not the *STPP* which contradicts Theorem (3.1).

Theorem 3.2— A horosphere in W has the *STPP*.

PROOF : The proof will also be carried out through contradiction. Assume that a horosphere H in W has not the *STPP*. This means that one (or both) of the following possibilities might happen

(i) there exists a geodesic sphere $S' \subset W$ such that $H \cap (W \setminus S')$ has more than two components;

(ii) there exists another horosphere $H' \subset W$ such that $H \cap (W \setminus H')$ has more than two components.

The case (i) is similar to (ii) in the proof of the last theorem. In case (ii) we have (Fig. 3). Using Theorem 2.3 there exists a pair of asymptotic geodesics γ_1 and γ_2 perpendicular to H and H'. According to the properties of W mentioned in section 1, H

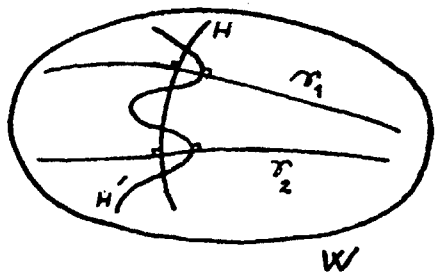

FIG. 3.

and H' should be equidistant. Since $H \cap H' \neq \phi$, then $H \equiv H'$ which is a contradiction and the proof of the theorem is now complete.

Similar to Corollary 3.1 we have

Corollary 3.2—A closed horodisc in W has the *STPP*.

4. APPLICATIONS

Before being involved in the following conclusions, one should remember that a complete, simply connected, Riemannian manifold with sectional curvature $K \leqslant 0$ has no focal points[6], and hence the Euclidean as well as hyperbolic spaces are good examples of this sort of manifolds.

In the following we show how definitions (i) and (ii) of Section 2 given by Bolton[3] tend to be the same ones (i) and (ii) of section 1 given by Banchoff[1] when the manifold W with no focal points is replaced by a Euclidean space E^{n+1}. Moreover, we demonstrate how Theorems 3.1 and 3.2 we have established lead to Theorem 1.1 of Banchoff[1] when the same replacement occurs. The idea is to show that a horosphere H_v in E^{n+1} is nothing but a hyperplane perpendicular to $v \in SE^{n+1}$ and a horodisc B_v in E^{n+1} is exactly the half-space with H_v as its boundary.

The way of doing so is to show how we obtain a horosphere H_v and its corresponding horodisc B_v in W for $v \in SW$ (i. e. we interpret geometrically the informations given throughout the second paragraph in section 2).

Actually, we draw a geodesic sphere $S_{\gamma_v(s)}(s)$ centered at $\gamma_v(s)$ with radius s, $s > 0$. Then if we let $s \to \infty$, $S_{\gamma_v(s)}(s)$ tends to be $H_v = b_v^{-1}(0)$. This might be expressed as saying that a horosphere is a geodesic sphere with infinite radius (see Fig. 4). The hypersurface $b_v^{-1}(t)$, $t \in \mathbf{R}$, is just a horosphere parallel to $H_v = b_v^{-1}(0)$ with distance $|t|$ apart. The horosphere H_{-v} is given in the same manner when considering the geodesic ray γ_{-v} and $S_{\gamma_{-v}(s)}(s)$.

In $E^{n+1}, \gamma_v(s)$ becomes a straight line and as $s \to \infty$, $S_{\gamma_v(s)}$ —which is a Euclidean sphere-becomes a hyperplane H_v. Moreover, $H_v = H_{-v}$. The horodisc $B_v = b_v^{-1}(0, \infty)$ is the half-space indicated in Fig. 5.

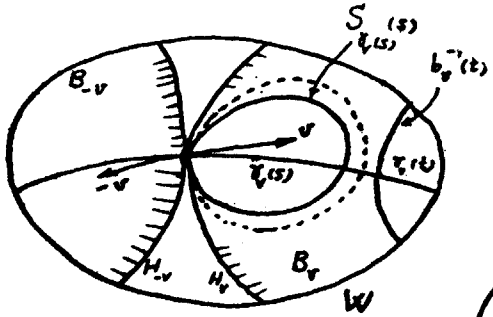

FIG. 4.

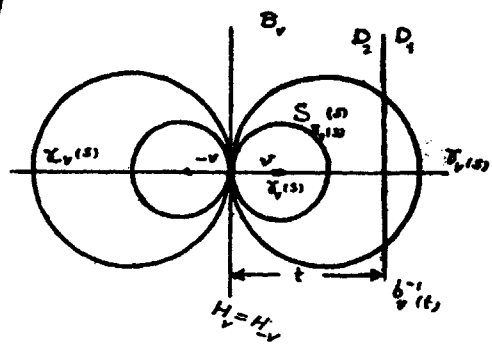

FIG. 5.

In Definitions (i) and (ii) of Section 2 and when $W = E^{n+1}$,

$$W \setminus b_v^{-1}(t) = E^{n+1} \setminus b_v^{-1}(t) = D_1 \cup D_2$$

where D_1 and D_2 are the same subsets given in the same definitions. Consequently, when $W = E^{n+1}$ we are led to the same definitions of *TPP* and *STPP* given by Banchoff[1]. According to the above discussion, Theorems 3.1 and 3.2 gives directly Theorem 1.1 when $W = E^{n+1}$.

In the light of the above two Theorems 3.1 and 3.2 and taking into account that hyperbolic space H^{n+1} is a Riemannian manifold without focal points we have that geodesic spheres and horospheres in hyperbolic space have the *STPP*. In this special case, the proof of the above mentioned Theorems 3.1 and 3.2 might be clear geometrically as follows. Consider the disc model of the hyperbolic space. In this model geodesic spheres are Euclidean spheres not touching the boundary of the disc while horospheres are those Euclidean spheres touching the boundary (see Fig. 6). Looking at the geodesic sphere S and horosphere H as Euclidean spheres in Euclidean space E^{n+1} they should have *STPP* in the Euclidean sense. So any geodesic sphere or horosphere in H^{n+1} have the *STPP* in hyperbolic sense. If we consider the half-space model of hyperbolic space instead of the disc model, similar discusion gives the same result.

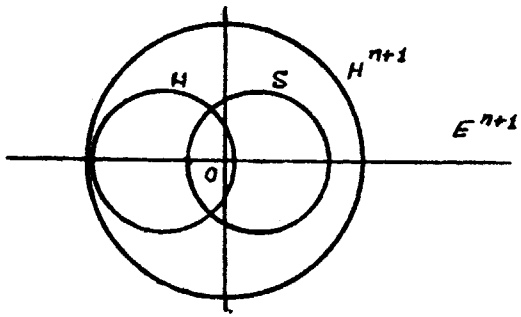

FIG. 6.

Corollary 4.1—Totally geodesic hypersurfaces of hyperbolic space have the *STPP*.

PROOF : Again if we consider the disc model H^{n+1} we have that totally geodesic hypersurfaces are hyperplanes of E^{n+1} passing through the origin 0 or hyperspheres perpendicular to the boundary of the disc. Clearly, these sorts of hypersurfaces have the *STPP* in Euclidean sense and consequently they have *STPP* in hyperbolic sense.

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