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ON GENERALIZED STIEFEL MANIFOLDS

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We determine the homotopy groups and the cohomology algebras of certain homogeneous spaces, which we call generalized Stiefel manifolds. Some of these homotopy groups classify the regular homotopy classes of certain sub-immersions of manifolds.

1. INTRODUCTION

A generalized Stiefel manifold is the space of $m \times n$ matrices of a fixed rank $k \leqslant \min(m, n)$. If $k = \min(m, n)$, then the space becomes a classical Stiefel manifold. These spaces were introduced by Milnor[5] for the study of immersions of manifolds, and Favaro[4] determined the integral homology groups of generalized Stiefel manifolds of rank 1 by the method of cellular decomposition. The purpose of this note is to determine the homotopy groups and the mod 2 cohomology algebras of generalized Stiefel manifolds.

Since the row (or column) space of an $m \times n$ matrix of rank k spans a k-plane, a generalized Stiefel manifold fibres over a Grassmannian with fibre a Stiefel manifold. This basic fact is the guideline of the calculations given here.

There are three kinds of generalized Stiefel manifolds according to whether the entries of matrices are real, complex, or quaternionic numbers. We shall consider here only the real case. The parallel results for other cases may be seen easily.

The homotopy groups of real generalized Stiefel manifolds are of special interest when one tries to prove the existence of a subimmersion (smooth map of constant rank) between certain manifolds. We shall show in paticular that the regular homotopy classes of subimmersions of fixed rank k of $S^k \times D^{n-k}$ into $\mathbf{R}^m$, where $k \leqslant n < m$, can be identified with the kth homotopy group of the real generalized Stiefel manifoid of $m \times n$ matrices of rank k. This will be an extension of the classical theorem of Smale on immersions of S^n into $\mathbf{R}^m$ (ref. 7).

2. HOMOTORY GROUPS

Let $M(m, n; k)$ be the generalized Stiefel manifold of $m \times n$ matrices of rank k with real entries. Then $GL(m, \mathbf{R}) \times GL(n, \mathbf{R})$ acts on the left of $M(m, n; k)$ by

(u, v), $x = u.\ x.\ v^t$, where v^t denotes the transpose of v, and the action is transitive. The subgroup of stability finding $\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \in M(m, n; k)$, I_k being the kth order identity matrix, is the group consisting of elements of the form

$$\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} (a^t)^{-1} & d \\ 0 & e \end{bmatrix}\right)$$

where $a \in GL(k, \mathbf{R})$, $c \in GL(m-k, \mathbf{R})$, $e \in GL(n-k, \mathbf{R})$, and 0 denotes a certain null matrix. It follows then, by a standard argument, that $M(m, n, k)$ has the homotopy type of the quotient space $(0(m) \times 0(n))/G$, where G is the subgroup consisting of elements of the form

$$\left(\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}, \begin{bmatrix} u & 0 \\ 0 & w \end{bmatrix}\right)$$

where $u \in 0(k)$, $v \in 0(m-k)$, $w \in 0(n-k)$.

Let $V_{m,k}$ be the Stiefel manifold of orthonormal k-frames in $\mathbf{R}^m$, and regard an element of $V_{m,k}$ as an $m \times k$ martix. Then $0(k)$ acts freely on the right of the product space $V_{m,k} \times V_{n,k}$ by the diagonal action (x, y), $g = (xg, yg)$. Let $V^k_{m,n}$ be the quotient space of $V_{m,k} \times V_{n,k}$ by equivalence under $0(k)$ and $p : V_{m,k} \times V_{n,k} \to V^k_{m,n}$ be the quotient map. Then the group $0(m) \times 0(n)$ is a transitive topological transformation group of $V^k_{m,n}$ under the action defined by $(a, b)\, p(x,y) = p(ax, by)$, and the subgroup of stability leaving $p\left(\begin{bmatrix} I_k \\ 0 \end{bmatrix}, \begin{bmatrix} I_k \\ 0 \end{bmatrix}\right)$ fixed is the group G considered above. This amounts to saying that $M(m, n; k)$ has the homotopy type of $V^k_{m,n}$.

Let $G_{m,k}$ be the Grassmann manifold of k-planes in $\mathbf{R}^m$. The construction of $V^k_{m,n}$ gives us a fibre bundle $V^k_{m,n} \to G_{m,k}$ with fibre $V_{n,k}$ and structure group $0(k)$, which is associated to the principal $0(k)$-bundle $V_{m,k} \to G_{m,k}$. If $n \geqslant m$, we have an $0(k)$-equivariant map $s : V_{m,k} \to V_{n,k}$ given by $s(x) = \begin{bmatrix} x \\ 0 \end{bmatrix}$. Therefore, by the correspondence between equivariant maps and cross-sections of an associated bundle, the fibre bundle $V^k_{m,n} \to G_{m,k}$ admits a cross-section. Therefore, by Steenrod[8], (Theorem 17.7), we get

Theorem 1— (i) If $n \geqslant m$, then

$$\pi_i\left(V^k_{m,n}\right) = \pi_i(V_{n,k}) \oplus \pi_i(G_{m,k}) \text{ for all } i \geqslant 1.$$

(ii) If $m \geqslant n$, then

$$\pi_i\left(V^k_{m,n}\right) = \pi_i(V_{m,k}) \oplus \pi_i(G_{n,k}) \text{ for all } i \geqslant 1.$$

Note that (ii) follows from (i) by interchanging the role of m and n. The case for $\pi_1\left(V^1_{2,2}\right)$ follows because $V^1_{2,2}$ has homotopy type of $S^1 \times S^1$.

3. Mod 2 Cohomology

Recall from Borel[2] that the algebra $H^*(V_{n,k}; \mathbb{Z}_2)$ has a simple system of generators $(x_{n-k},\ldots,x_{n-1})$ where degree $x_j = j$. We denote the simple system of generators of $H^*(V_{n+1,k}; \mathbb{Z}_2)$ by $(y_{n-k+1}, \ldots, y_n)$ and that of $H^*(V_{n+1,k+1}; \mathbb{Z}_2)$ by $(z_{n-1}, \ldots, z_n)$.

Lemma 1—If $i : V_{n,k} \to V_{n+1,k}$ is the natural inclusion, then the elements x_{n-k}, $i^*(y_{n-k+1})$, $\ldots$, $i^*(y_{n-k+r})$ generate a subalgebra of $H^*(V_{n,k}; \mathbb{Z}_2)$ containing the elements $x_{n-k}, \ldots, x_{n-k+r}$, where r is a number such that $1 \leqslant r \leqslant k-1$.

Proof : First note that the map i is the composition of the inclusion $f : V_{n,k} \to V_{n+1,\,k+1}$ followed by the projection $g : V_{n+1,\,k+1} \to V_{n+1,k}$. By the Serre exact sequence of the fibre bundle $V_{n+1\,k+1} \to S^n$ with fibre $V_{n,k}$, $f^* : H^q(V_{n+1,k+1}; \mathbb{Z}_2) \to H^q(V_{n,k}; \mathbb{Z}_2)$ is a monomorphism for $q < n-1$. But for $q \leqslant n-1$, $H^q(V_{n+1,k+1}; \mathbb{Z}_2)$ and $H^q(V_{n+1,k+1}; \mathbb{Z}_2)$ are isomorphic as finite dimensional vector spaces. Therefore f^* is an isomorphism for $q \leqslant n-1$. Again, by Borel[2] $g^*(y_j) = z_j$ for $n-k+1 \leqslant j \leqslant n-1$. These make the proof of the lemma quite clear.

Lemma 2—The systtem of local coefficients $H^*(V_{n,k}; \mathbb{Z}_2)$ in the fibre bundle $V^k_{m,n} \to G_{m,k}$ is simple.

Proof : Assume inductively that (for a fixed k and any n) the generators $x_{n-k},\ldots,$ x_{n-k+r} of the cohomology of the fibre $V_{n,k}$ over some point $x \in G_{m,k}$ are invariant under the action of the fundamental group $\pi_1(G_{m,k}, x)$, for some $r < k-1$. The induction may be started, as $x_{n-k} \in H^{n-k}(V_{n,k}; \mathbb{Z}_2) \approx \mathbb{Z}_2$ whose only automorphism is the identity. Now we have a bundle map

$$\begin{array}{ccc} V^k_{m,n} & \to & T^n_{m,n+1} \\ \searrow & & \swarrow \\ & G_{mk} & \end{array}$$

induced by the inclusion of the $0(k)$-equivariant subspace $V_{n,k}$ in $V_{n+1,k}$. This gives a homomorphism of the local systems, which at the point x is $i^* : H^*(V_{n+1,k}; \mathbb{Z}_2) \to H^*(V_{n,k}; \mathbb{Z}_2)$. Now, by the inductive hypothesis, the generators $y_{n+k+1},\ldots, y_{n-k+1+r} \in H^*(V_{n+1,k}; \mathbb{Z}_2)$ are invariant under the action $\pi_1(G_{m,k}, x)$. Therefore the elements $i^*(y_{n-k+1}), \ldots,$ $i^*(y_{n-k+1+r})$ are invariant in $H^*(V_{n,k}; \mathbb{Z}_2)$. Hence, by Lemma 1, $x_{n-k+1+r}$ is also invariant. This completes the inductive step, and so the proof of the lemma.

Lemma 3—The generators $x_{n-k}, \ldots, x_{n-1} \in H^*(V_{n,k}; \mathbb{Z}_2)$ are transgressive in the fibre bundle $V^k_{m,n} \to G_{m,k}$.

PROOF : Noting that the generator x_{n-k} of lowest degree is clearly transgressive, we simply persue an induction similar to that of Lemma 2.

Lemma 4—All transgressions in the fibre bundle $V^k_{m,n} \to G_{m,k}$ are trivial when $n \geqslant m$.

PROOF : Observe that as the fibre bundle admits a cross-section, p^* is monomorphism, where p is the bundle projection.

With these lemmas in hand, we see that spectral sequence of the fibre bundle $V^k_{m,n} \to G_{m,k}$ collapses when $n \geqslant m$. This gives

Theorem 2— (i) If $n \geqslant m$, then

$$H^*\left(V^k_{m,n} \; ; \mathbb{Z}_2\right) = H^*(V_{m,k}; \mathbb{Z}_2) \otimes H^*(G_{m,k}; \mathbb{Z}_2).$$

(ii) If $m \geqslant n$, then

$$H\left(V^k_{m,n} \; ; \mathbb{Z}_2\right) = H^*(V_{m,k}; \mathbb{Z}_2) \otimes H^*(G_{n,k}; \mathbb{Z}_2).$$

Note that (ii) follows from (i) by interchanging m and n.

4. APPLICATIONS

Let X and Y be smooth manifolds of dimensions n and m respectively and k be a fixed constant $\leqslant \min(m, n)$. A smooth map $X \to Y$ is called a k-subimmersion if it has rank k at each point of X. Let $\mathrm{Sim}_k(X, Y)$ be the space of such k-subimmersions with the C^∞ topology of compact convergence. On the other hand, call a map between tangent bundles $TX \to TY$ a k-map if it carries each fibre linearly into a fibre, and if its restriction to each fibre has rank k. Let $\mathrm{Hom}_k(TX, TY)$ be the space of such k-maps with the compact open topology. Then, according to Phillips[6], if X has no closed component and has no handle of index $> k$, the differential map d : $\mathrm{Sim}_k(X, Y) \to \mathrm{Hom}_k(TX, TY)$ is a weak homotopy equivalence.

Henceforth we shall assume that $y = \mathbf{R}^*$. We may identify the space $\mathrm{Hom}_k(TX, TY)$ with the space of cross-sections of a certain fibre bundle $p : E \to X$. Indeed, consider the fibre bundle $q : E \to X \times Y$ whose fibre over the point $(x, y) \in X \times Y$ is the space of k-maps between tengent spaces $(TX)_x \to (TY)_y$, and define p as the composition of q followed by the left projection. Now let $f : X \to Y$ be an arbitrary smooth map, and $q' : E' \to X$ be the pull back of $q : E \to X \times Y$ by the map $X \to X \times Y$ given by $x \mapsto (x, f(x))$. Then the set of k-maps $TX \to TY$ which induce f on the base corresponds bijectively with the set of cross-sections of the bundle $q' : E' \to X$. Therefore, for f to be homotopic to a k-subimmersion it is necessary and sufficient that there is a cross-section of $q' : E' \to X$. As the fibre of this bundle is homotopic to $V^k_{m,n}$, one

meets obstructions which are elements of the cohomology groups $H^i\left(X; \pi_{i-1}\left(V^k_{m,n}\right)\right)$. As the space $V^k_{m,n}$ is not n-simple, one should expect in the above cohomology a system of local groups as coefficients which display the action of $\pi_1\left(V^k_{m,n}\right)$ on $\pi_{i-1}\left(V^k_{m,n}\right)$. If X is simply connected, this complication of twisted coefficients can be avoided by using the obstruction theory of Barcus[1] as developed by Hermann[3]. The main idea here is to define the obstruction cocycles and difference cochains in terms of certain global homotopy properties of a fibre bundle. These definitions fit well into the classical obstruction theory. In particular we may retrieve a classification theorem of Steenrod[8] [§ 37.5] in the following form : if $E \to X$ admits a cross-section s_0, and if

$H^i\left(X; \pi_i\left(V^k_{m,n}\right)\right) = 0$ for $i = 0, 1, \ldots, n-1$, and

$H^{i-1}\left(X; \pi_i\left(V^k_{m,n}\right)\right) = 0$ for $i = 0, 1, \ldots, n-1$, where H^0 is understood to be the reduced cohomology, then the assignment of the difference cochain $d(s, s_0)$ to each cross-section s sets up a $1-1$ correspondence between homotopy classes of based cross-sections are elements of $H^n\left(X; \pi_n\left(V^k_{m,n}\right)\right)$. We omit the obvious details which may be seen easily.

Now if $X = S^k \times D^{n-k}$, $k < n \leqslant m$, the existence of the cross-section s_0 follows trivially, since S^k immerses in $\mathbf{R}^m$, and therefore using the k-th skeleton of X (there being no higher obstructions) in the above arguments, We get :

Theorem 3—If $k < n \leqslant m$, the set of regular homotopy classes of based k-submersions of $S^k \times D^{n-k}$ into $\mathbf{R}^m$ corresponds bijectively with the elements of the homotopy group $\pi_k\left(V^k_{m,n}\right)$.

If $k = n < m$, the theorem is the classical theorem of Smale[7].

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