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HOPFIAN RINGS AND HOPFIAN MODULES

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In this paper we generalize Baumslag's notion of Hopficity for groups to rings and modules. If R is a Hopfian Boolean ring then the space X of all the maximal ideals of R equipped with the hull-kernel topology is co-Hopfian in the sense that every continuous one-one map of X into itself is a homeomorphism. We have the complete description of Hopfian free modules over commutative rings, Hopfian modules over semisimple artinian rings and Hopfian projective modules over right semiperfect rings.

Baumslag[3] has introduced the notion of Hopfian groups. We introduce, in this paper, the notions of Hopfian rings and Hopfian modules and study their properties over special types of rings. If R is a Boolean ring and X is the space of all the maximal ideals of R equipped with the hull-kernel topology and if R is Hopfian then X is co-Hopfian in the sense that every injective continuous map of X into X is a homeomorphism. Any commutative ring R with identity is Hopfian as an R-module. Over a commutative ring with identity Hopfian free modules are just the free modules of finite rank. If R is a right semiperfect ring, Hopfian projective right R-modules are precisely the finitely generated projective right R-modules. For a semisimple artinian ring R, Hopfian right R-modules are just the right R-modules of finite length.

Throughout this paper, by a ring R we mean an associative ring with identity and by an R-module we mean an unitary right R-module.

We first recall that Baumslag[3] defined a group G to be 'Hopfian' if every homomorphism of G onto G is an automorphism of G.

We now generalize this concept of Hopficity to rings and modules.

HOPFIAN RINGS

Definition 1—A ring R is said to be 'Hopfian' if every ring homomorphism of R onto R is an automorphism of R.

Clearly every simple ring, in particular, any field is Hopfian.

Proposition 2—(i) Every right (or left) noetherian ring is Hopfian.

(ii) Every right (or left) artinian ring is Hopfian.

(iii) Every commutative integral domain, in which every nonzero prime ideal is maximal, is Hopfian.

(iv) If R is a ring such that the polynomial ring $R[X]$ over R in an indeterminate X is Hopfian, then itself if Hopfian.

(v) If R is a ring such that the ring R_n, for some $n \geqslant 1$, of all $n \times n$ matrices over R is Hopfian, then R is Hopfian.

PROOF : (i), (iii), (iv) and (v) are easy verifications and (ii) follows from (i).

We do not know whether the converses of (iv) and (v) of the above proposition are true.

We now show that if R is a Hopfian Boolean ring then the space X of all the maximal ideals of R equipped with the hull-kernel topology is co-Hopfian in the sense that every continuous one-one mapping of X into X is a homeomorphism.

Stone[8] has proved that if R is a Boolean ring and X is the space of all the maximal ideals of R equipped with the hull-kernel topology, then X is a compact, Hausdorff and totally disconnected space. Moreover, R is isomorphic to the ring of all the clopen subsets of X.

Theorem 3—Let R be a Boolean ring and let X be the space of all the maximal ideals of R equipped with the hull-kernel topology. If R is Hopfian then X is co-Hopfian in the sense that every continuous injective map of X into X is a homeomorphism.

Before proving the theorem we prove the following lemma.

Lemma 4—Let X be a compact, Hausdorff and totally disconnected space. Let Y be a closed subspace of X. If A is a clopen subset of Y then there exists a clopen subset B of X such that $A = B \cap Y$.

PROOF : Now A is closed in Y and Y is closed in X and hence, A is closed in X. Since X is compact, A is also compact. Since X is compact, Hausdorff and totally disconnected, the clopen subsets $\{B_\alpha, \alpha \in I\}$ form a base for the topology of X, by IX. 6.4 of Bourbaki[4]. Then clearly $\{B_\alpha \cap Y : \alpha \in I\}$ will be a base for the subspace topology of Y. Since A is open in Y, $A = \bigcup_{\alpha \in I'} (B_\alpha \cap Y)$ for some subset I' of I. Since A is compact, there exist $\alpha_1, \ldots, \alpha_n$ in I' such that $A = \bigcup_{i=1}^{n} (B_{\alpha_i} \cap Y) = B \cap Y$, where $B = \bigcup_{i=1}^{n} B_{\alpha_i}$ is clopen in X.

Proof of the Theorem—Let $f : X \to X$ be a continuous one-one map of X into X. Since X is compact and hence $f(X)$ is also compact and hence $f(X)$ is closed in X since X is Hausdorff.

Let A be a clopen subset of X. Since X is compact, Hausdorff, f is a homeomorphism of X onto $f(X)$, and hence, $f(A)$ is clopen in $f(X)$. Then, by Lemma 4, there exists a clopen subset B of X such that $f(A) = B \cap f(X)$.

By Stone's theorem, stated above, R is isomorphic to the ring S of all the clopen subsets of X, where the ring operations in S are given by $A + B = A \Delta B$ and $AB = A \cap B$. Now define $f^* : S \to S$ by $f^*(A) = f^{-1}(A)$ for each A in S. Clearly f^* is a ring homomorphism. We claim that f^* is onto. Let $A \in S$ so that A is clopen in X. Then, by the above paragraph, there exists a clopen set B in X such that $f(A) = B \cap f(X)$. Then $A = f^{-1} f(A) = f^{-1}(B \cap f(X)) = f^{-1}(B) \cap X = f^{-1}(B)$ since f is one-one. So $B \in S$ and $f^*(B) = A$. Thus f^* is onto. Since R is Hopfian, so is S. Hence f^* is an automorphism of S. We now prove that f is onto. Suppose the contrary. Let $x \in X \setminus f(X)$. Since $f(X)$ is closed in X, $X \setminus f(X)$ is open in X. So there exists a clopen set A in X such that $x \in A \subseteq X \setminus f(X)$. Then $f^{-1}(A) = \phi = f^{-1}(\phi)$ which implies that $f^*(A) = f^*(\phi)$. Since $A \neq \phi$ this contradicts the fact that f^* is one-one. So f is onto and hence is a homeomorphism.

We do not know whether the converse of the above theorem is true.

Hopfian Modules

Definition 5—An R-module M is said to be Hopfian if every ependomorphism of M is an automorphism of M.

Clearly every simple R-module is Hopfian.

Proposition 6—(i) Every noetherian R-module is Hopfian.

(ii) If R is a right noetherian ring then every finitely generated R-module is Hopfian.

Proof : (i) is an easy verification and (ii) follows from (i).

It is easy observation that every direct summand of a Hopfian R-module is Hopfian. But we do not know whether every direct sum (even finite) of Hopfian modules is Hopfian.

Remark 7 : We do not know whether the Hopficity is a hereditary property. But Hopficity is not preserved under taking factor modules. For example, the additive group Q, of all rational numbers, is clearly Hopfian as a Z-module. But the 2-Prüfer group $Z(2^\infty)$ is not Hopfian as a Z-module since the multiplication by 2 is an ependomorphism of $Z(2^\infty)$ but not an automorphism of $Z(2^\infty)$. Note that $Z(2^\infty)$ is a factor module of Q.

Proposition 8—If R is a commutative ring then R is Hopfian as an R-module.

PROOF : If $f: R \to R$ is an endomorphism of R as an R-module and $f(1) = a \in R$, then f is given by left multiplication by a. If f is an epimorphism then a will be a unit in R which implies that f is an automorphism.

More generally, we have

Theorem 9—A ring R is Hopfian as an R-module if and only if no right zero divisor in R is a left or a right unit in R.

PROOF : *Only if* : Suppose R is Hopfian. Let $y \in R$ be a right zero divisor in R. Then there is an $r \neq 0$ in R such that $yr = 0$ Then clearly y is not a left unit in R. Suppose y is a right unit in R. Let $z \in R$ be such that $yz = 1$. Then clearly left multiplication by y is an ependomorphism of R. Then, by the Hopficity of R, the left multiplication by y is an automorphism of R which contradicts the fact that $yr = 0$ and $r \neq 0$. So y is not a right unit in R.

If : Follows by noting that every R-endomorphism α of R is given by left multiplication by some element y of R and α is onto if and only if y is a right unit in R.

Remark 10 : If an R-module is Hopfian then its injective hull need not be Hopfian. For example, the cyclic group $Z(3)$ of order 3 is trivially Hopfian but its injective hull $Z(3^\infty)$ is not Hopfian since the multiplication by 3 is an ependomorphism of $Z(3^\infty)$ but it is not an automorphism of $Z(3^\infty)$. But the converse is true in some special cases.

Proposition 11—Let R be a commutative integral domain. Let M be a torsion-free R-module such that its injective hull $E(M)$ is Hopfian. Then M is Hopfian.

PROOF : Let $f: M \to M$ be an ependomorphism of M. Then f extends to an endomorphism g of $E(M)$. Since M is torsion-free, so is $E(M)$ and hence, Im (g) is torsion-free. Since $E(M)$ is divisible, Im (g) is also divisible. Now Im (g) is injective, by VII, Proposition 1.3 of Cartan and Eilenberg[5]. Thus Im (g) is a direct summand of $E(M)$. Since M is essential in $E(M)$ and $M = f(M) \subseteq \text{Im}(g) \subseteq E(M)$, Im (g) is also essential in $E(M)$. So Im $(g) = E(M)$ proving that g is onto. Since $E(M)$ is Hopfian, g, and hence, f is an automorphism. Thus M is Hopfian.

We now investigate when free modules are Hopfian.

Proposition 12—If a free R-module is Hopfian then it is of finite rank.

PROOF : Let F be a free R-module of infinite rank so that $F = (\bigoplus_{n=1}^{\infty} R_n) \oplus G$ where $R_n = R$ for each $n \geq 1$ and G is a submodule of F. Define an endomorphism α of F by

$\alpha(R_1) = 0$, $\alpha(R_n) = R_{n-1}$ for $n > 1$ and $\alpha \mid G = Id_G$.

Clearly α is an epimorphism but not an automorphism. So F is not Hopfian. Hence the proposition follows.

Remark 13 : A free module of finite rank need not be Hopfian. Actually something more is true, namely, every ring R need not be Hopfian as an R-module.

Example—Let K be a field and let $V = \bigoplus_{n=1}^{\infty} K_n$, where $K_n = K$ for each n, be a countably infinite dimensional vector space over K. Let R be the ring of all linear operators of V. We show that R is not Hopfian as an R-module. We define $\alpha, \beta, \gamma \in R$ as follows.

$$\alpha(K_1) = 0, \alpha(K_n) = K_{n-1} \text{ for } n > 1;$$

$\beta(K_n) = K_{n+1}$ for each $n \geqslant 1$; and

γ is the projection of V onto K_1 with kernel $\bigoplus_{n \geqslant 2} K_n$.

Then clearly $\alpha\beta = 1$ and $\alpha\gamma = 0$. So α is a right zero divisior in R and also a right unit in R. Then, by Theorem 9, R is not Hopfian as an R-module.

The converse of Proposition 12 is true when R is commutative.

Theorem 14—If R is a commutative ring then a free R-module is Hopfian if it is of finite rank.

PROOF : Follows from Proposition 12 and Exercise 4, p. 169 of Jacobson[7].

Corollary 15—If R is a commutative ring then every finitely generated projective R-module is Hopfian.

PROOF : Since a finitely generated projective R-module is a direct summand of a free R-module of finite rank, the corollary follows from the above theorem.

We now characterize the Hopfian vector spaces and Hopfian modules over a semisimple artinian ring.

Theorem 16—(i) If R is a semisimple artinian ring then an R-module M is Hopfian if and only if M has finite length.

(ii) A vector space over a field F is Hopfian if and only if V is finite dimensional.

PROOF : (i) Since every module over a semisimiple artinian ring is semisimple (p. 356 and Chapter 4, Exercise 6 of Faith[6]), the proof for 'only if' similar to that of Proposition 12 and 'if' follows from Proposition 6 (i) by noting that every module of finite length is noetherian (Chapter III, Theorem 21 of Zariski and Samuel[9]).

(ii) Follows from (i).

A result in line with the above assertions is the following.

Proposition 17—If R is a commutative ring the matrix ring $S = R_n$, of all $n \times n$ matrices over R, for $n > 0$, is Hopfian as an S-module.

PROOF : An S-endomorphism f of S is given by left multiplication by an element A of S. If f is onto, there is a B in S such that $AB = I$. So A is non-singular and hence f is an automorphism of S.

We conclude our study of Hopfian modules by describing the Hopfian projective modules over a right semiperfect ring[2].

Theorem 18—Let R be a right semiperfect ring. Then a projective R-module P is Hopfian if and only P is finitely generated.

PROOF : *Only if* : Since R is right semiperfect and P is projective, it follows, by Theorem 27.11 of Anderson and Fuller[1], that there exist sets $A_1,\ldots,A_n$ such that P is isomorphic to $(e_1 R)^{(A_1)} \oplus \ldots \oplus (e_n R)^{(A_n)}$, where $e_1,\ldots,e_n$ is a basic set of primitive idempotents of R. Now, by the Hopficity of P, each of the sets $A_1,\ldots,A_n$ must be finite and hence P is finitely generated.

If : Let J be the Jacobson radical of R. Since R is right semiperfect, R/J is semisimple artinian (Bass[2], Theorem 2.1). Since P/PJ is finitely generated R/J-module, with the induced structure, it follows, by Proposition 6 (ii), that P/PJ is Hopfian as an R/J-module. Now let α be an R-ependomorphism of P. Since PJ is invariant under α, α induces an R-ependomorphism (and hence an R/J-ependomorphism) α^* of P/PJ such that $\pi\alpha = \alpha^*\pi$, where $\pi : P \to P/PJ$ is the canonical R-epimorphism. Since P/PJ is Hopfian as an R/J-module, α^* is an automorphism of P/PJ. Then $K = Ker(\alpha) \subseteq PJ$. Since P is finitely generated R-module, PJ is small in P, by Nakayama lemma. Since $\mathrm{Im}(\alpha) = P$ is projective, K is a direct summand of P. Thus $K = 0$ proving that α is an automorphism of P and hence P is Hopfian.

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