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TOPOLOGICAL GROUPS ACTING ON SETS

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In this paper, the topology on a set acted on by a topological group is defined as a final topology with respect to a set of maps. We mainly prove that this topology is a regular topology. We also prove that when a first countable T_0-topological group acts on a set X, then X is a G_δ-diagonal space.

INTRODUCTION

A topology on a set can be defined by several methods e.g., via closures, interiors, nets and filters, etc. Sometimes a topology constructed out of given topologies may be useful.

In Patronis[4] a topology τ_p is introduced on a set X acted on by a topological group G through a closure operator. In this paper, variety of useful characterizations of this topology are considered. We show that the topology τ_p coincides with the final topology τ_f introduced on X by a set of functions $\{\hat{g}\}$;

$$\hat{g} : G \to X.$$

Also we show that if a first countable T_0-topological group acts on a set X then the space X has a G_δ-diagonal.

DEFINITIONS AND PRELIMINARIES

Let G be a topological group and let X be any non-empty set. The group G acts as an abstract group on X if to each $g \in G$ and each $x \in X$, there corresponds a unique element gx of X such that

(i) $g_2 (g_1 x) = (g_2 g_1) x$ for every $g_1, g_2 \in G$ and for every $x \in X$.

(ii) $1x = x$

When G acts on a set X, two families of functions can be defined as follows :

To each $g \in G$, we define $\hat{g} : X \to X$.

$$\hat{g}(x) = gx.$$

To each $x \in X$, we define a function

$$\hat{x} : G \to X,$$

$$\hat{x}(g) = gx.$$

Definition 1 (Patronis[4])—A topology τ_p on X is introduced as follows :

For any $x \in X$ and any subset A of X, $x \in \bar{A}$ iff for every neighborhood Γ of the identity 1 of G, there exists $g \in \Gamma$ with $gx \in X$. Under this topology the functions $\{\hat{g}\}$ are homeomorphisms.

The topology τ_p can be defined by means of the interior operator as follows :

$x \in \text{Int } A$ iff there exists a neighborhood Γ of the neighborhood system of identity $\mathscr{N}$ of G such that $\Gamma x \subset A$.

It follows that for any x of X and any Γ of $\mathscr{N}$, Γx is a neighborhood of x.

A subset A of X is open if for every x of A there exists Γ_x of $\mathscr{N}$ so that $\bigcup_{x \in A} \Gamma_x = A$.

Characterizations of the Topology Introduced on X

Proposition 2– For any subset U of X and any symmetric neighborhood Γ of 1 $\bar{U} \subset \Gamma U$.

Proof : To prove proposition, we prove that $x \notin \Gamma U$ implies $x \notin \bar{U}$.

For $g \in \Gamma$, $g^{-1} \in \Gamma$. Therefore, $x \notin \Gamma U$ i.e. $g^{-1}(gx) \notin \Gamma U$. Thus $gx \notin U$ for otherwise, from symmetry of Γ, implies $x \in \Gamma U$, a contradiction.

Proposition 3—For any subset U of X, $\bar{U} = \bigcap_{\Gamma \in \mathscr{N}} \Gamma U$.

Proof : From Proposition 2, $\bar{U} \subset \Gamma U$. To prove the converse inclusion, let $x \in \bigcap \Gamma U$. Then $x \in \Gamma U$ for every Γ, i.e. $x = \gamma u$ for some $\gamma \in \Gamma$ and $u \in U$. If Γ is symmetric, then $\gamma^{-1} \in \Gamma$ and hence

$$\gamma^{-1} x = u \in U \text{ i.e. } x \in \bar{U}.$$

Corollary 4—For any $x \in X$ and any symmetric neighborhood Γ of $\mathscr{N}$

$$\overline{\Gamma x} = \Gamma x.$$

Indeed, $\overline{\Gamma x} = \bigcap_{K \in \mathscr{N}} Kx$

i.e. $\overline{\Gamma x} \subset \Gamma x$, which implies

$$\overline{\Gamma x} = \Gamma x.$$

Corollary 5—The topology τ_p induced by the action of a topological group is a regular topology.

Let X be endowed with the final topology τ_f induced by the family of functions $\{\hat{x}\}$,

$$\hat{x} : G \to X, \text{ where } \hat{x}(g) = gx.$$

It is easy to see that $\hat{x}$ is onto.

Theorem 6—The topology τ_p introduced through a closure operator coincides with the topology τ_f defined by the set of functions $\{\hat{x}\}$.

PROOF : For any $x \in X$, the function

$\hat{x} : G \to (X, \tau_p)$ is continuous, since for any $gx \in X$ and any neighborhood Γx of gx, where $g \in G$, we have $\hat{x}(\Gamma) = \Gamma x$.

Therefore, $\tau_p \leqslant \tau_f$.

To prove the converse, we shall show that $\bar{A}_{\tau_p} \subset \bar{A}_{\tau_f}$ for any subset A of X and consequently, $\tau_f \leqslant \tau_p$.

Let $x \in \bar{A}_{\tau_p}$ and U a neighborhood of x in τ_f. Then $\hat{x}^{-1} U$ is a neighborhood of identity 1 of G. Since $x \in \bar{A}_{\tau_p}$, then there exists $h \in \hat{x}^{-1} U$ so that $hx \in A$.

Also $hx \in U$ implies $A \cap U \neq \varphi$. Therefore, $x \in \bar{A}_{\tau_f}$ i.e. $\bar{A}_{\tau_p} \subset \bar{A}_{\tau_f}$ and the theorem follows.

Proposition 7—For any $x \in X$, orbit x is an open set.

The proof follows directly from

$$\text{Im } \hat{x} = \hat{x}G = Gx = \text{orbit } x.$$

Also we have

Propositions 8—Let X be a T_1-space. Then

(i) $\hat{x}^{-1}(x) = \text{stab } (x)$ is a closed subgroup

(ii) $G_Y = \{h \in G : hy = y, \forall y \in Y\}$ is closed subgroup.

(iii) If Y is an orbit of X, then G_Y is a normal closed and open subgroup of G.

Corollary 9—Simple groups act transitively.

Let H be a subgroup of a group G and τ_H denotes the topology on X induced by restriction of the action on H.

A topological space X is said to have a G_δ-diagonal if the diagonal in $X \times X$ is a G_δ-set. The G_δ-diagonal concept plays a useful role in characterizing metrizable or de velopable spaces[1,3]. The following lemma due to Ceder gives an equivalent form of the G_δ-diagonal property.

Lemma 10 (Ceder[2])—A space X has a G_δ-diagonal if and only if there is a sequence $\{Q_n\}_1^\infty$ of open covers of X such $\bigcap_{n=1}^{\infty} St(x, Q_n) = \{x\}$.

Theorem 11—Let a first countable T_0-topological group G act on a set X. Then X is a space with G_δ-diagonal.

PROOF : Let $\{\Gamma_i\}_{i=1}^\infty$ be a fundamental system of symmetric open neighborhoods of the identity 1_G of G and let, $Q_i = \{\Gamma_i x\}_{x \in X}$ be an open cover of X.

The star $St(x, Q_i)$ of the singleton $\{x\}$ in the cover Q_i is given by

$$St(x, Q) = \{\cup\ \Gamma_i y : \text{there exists } g \in \Gamma_i \text{ with } gy = x\}$$

$$= \bigcup_{g \in \Gamma_i} \Gamma_i g^{-1} x = \Gamma_i \Gamma_i^{-1} x.$$

Clearly,

$$x \in \Gamma\ st(x, Q_i) = P.$$

To show that P consists of the singleton $\{x\}$. Let $y \in P$, $y \neq x$. Since G is T_1-space then there exists a symmetric neighborhood Γ of 1_G, So that $\Gamma x = \Gamma^{-1} x$ misses y.

Let V be a symmetric neighborhood of 1_G so that $V^2 \subset \Gamma$, $VV^{-1}x$ misses y and hence

$$\cap\ St(x, Q_i) = \cap\ \Gamma_i \Gamma_i^{-1} x = \{x\}.$$

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