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ON CERTAIN EXTREMAL PROBLEMS

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The problems of finding the largest values of the moduli of the coefficients of generalized polynomials by four systems of functions are solved.

(a) The problem of D. Mendeleev about finding the largest values of moduli of coefficients of the algebraic polynomials of degree n, whose moduli are bounded by 1 on the segment [0, 1], was solved partly by Markov[10] and completely by Bernstein[1]. In that connection Schwartz[11] considered the following more general problem. "If $\mu_0, \mu_1, \ldots, \mu_n$ are real, non-negative and different numbers, then what are the largest values of moduli of the coefficients $a_0, a_1, \ldots, a_n$ of the generalized polynomial $P(x) = a_0 x^{\mu_0} + a_1 x^{\mu_1} + \ldots + a_n x^{\mu_n}$ bounded by 1 on the segment [0, 1] ?

Incidentally, the condition $\max_{0 \leqslant x \leqslant 1} | P(x) | \leqslant 1$ may be replaced by a weaker condition $\|P\|_{L^p(0,1)} \leqslant 1$ $(p > 1)$ and therefore μ_ν must be $> -1/p$ instead of $\geqslant 0$".

In the case $p = 2$, Schwartz[11] obtained the exact values of these maxima and built the corresponding extremal generalized polynomials, and for $p \geqslant 1$ $(p \neq 2)$ he obtained the exact asymptotic values of these maxima.

(b) In this article we deal with similar extremal problems for generalized polynomials in different systems of functions. In general form it may be formulated as follows :

"If $\{x_j\}_1^n$ $(n \geqslant 1)$ is a linearly independent system of elements from the Hilbert space H then : what are the largest values of coefficients $d_1, d_2, \ldots, d_n$ of the generalized polynomial $P = d_1 x_1 + d_2 x_2 + \ldots + d_n x_n$, whose norm is bounded by 1 ?

More concretely, it is required to find

$$N_H\left(\{x_j\}_1^n ; k\right) = \max_{\|P\| \leqslant 1} \{ | d_k | \} \, (1 \leqslant k \leqslant n)". \qquad \ldots(0.1)$$

The solution of this problem is connected with the construction of a specific system of elements of H which is biorthogonal to $\{x_j\}_1^n$. Namely, let $\{y_{nk}\}_1^n$ be a system of elements from the linear envelope of $\{x_j\}_1^n$, which is biorthogonal to it :

$$y_{nk} = \sum_{j=1}^{n} c_{jk}\, x_j \ (1 \leqslant k \leqslant n) \qquad \ldots(0.2)$$

$$(x_j, y_{nk}) = \delta_{kj} \ (1 \leqslant k, j \leqslant n)$$

where δ_{kj} is the Kronekker symbol. Then for generalized polynomial $P = d_1 x_1 + d_2 x_2 + \ldots + d_n x_n$ with $\|P\| \leqslant 1$ we will have $| d_k | = | (P, y_{nk}) | \leqslant \|P\| \, \|y_{nk}\| \leqslant \| y_{nk} \|$ $(1 \leqslant k \leqslant n)$. And if

$$P^*_{nk} = y_{nk}/\|y_{nk}\| = d^*_1 x_1 + d^*_2 x_2 + \ldots + d^*_n x_n \ (1 \leqslant k \leqslant n) \qquad \ldots(0.3)$$

then

$$| d^*_k | = | (P^*_{nk}, y_{nk}) | = \|y_{nk}\| \ (1 \leqslant k \leqslant n).$$

From the above it is clear, that

$$N_H \left(\{x_j\}_1^n ; k \right) = \|y_{nk}\| \ (1 \leqslant k \leqslant n) \qquad \ldots(0.4)$$

moreover, the equality (0.3) defines the extremal generalized polynomial, for which in (0.1) the maximum is attainable.

Thus, the problem is reduced to generating a system $\{y_{nk}\}_1^n$ with properties (0.2).

For different systems of rational functions and for generalized systems of Müntz-Szasz the biorthogonal systems with the mentioned properties were built explicity by Djrbashian[3-6]. In the case of four concrete systems of functions we arrive at the realization of the previously discussed problem; namely, we find the exact values of appropriate maxima (0.1) and extremal polynomials. From what is mentioned above it is clear, that the problem is reduced to a simple calculation of norms of the functions of the corresponding biorthogonal systems of Djrbashian.

§ 1. The systems $\{\tau_j (z)\}_1^n$ and $\{\Omega_{nk} (z)\}_1^n$, are biorthogonal on the real axes $(-\infty, +\infty) = \mathbf{R}$.

(a) Let $\{\Lambda_j\}_1^n$ $(\operatorname{Im} \Lambda_j > 0)$ be any bounded sequence of complex numbers from the upper half-plane and s_k and $p_k (n)$ $(1 \leqslant k \leqslant n)$ are numbers of the appearance of the number Λ_k in the sequences $\{\Lambda_j\}_1^k$ and $\{\Lambda_j\}_1^n$ respectively. Obviously $1 \leqslant s_k \leqslant p_k (n) \leqslant n$. The system of rational functions $\{\tau_j (z)\}_1^n$ is defined as follows :

$$\tau_j (z) = \frac{(s_j - 1)!}{(z - \Lambda_j)^{s_j}} \ (j = 1, 2, \ldots, n) \qquad \ldots(1.1)$$

(see Djrbashian[6]).

Define by

$$B_n(z) = \prod_{j=1}^{n} \frac{z - \Lambda_j}{z - \bar{\Lambda}_j} \qquad \ldots(1.2)$$

the finite Blaschke product in the upper half-plane and again following Djrbashian[6], consider the system of functions

$$\Omega_{nk}(z) = \frac{B_n(z)}{(s_k - 1)!} \sum_{\nu=0}^{p_k(n)-s_k} \frac{a_{n\nu}(\Lambda_k)}{(z - \Lambda_k)^{p_k(n)-s_k-\nu+1}} \quad (1 \leqslant k \leqslant n) \qquad \ldots(1.3)$$

where

$$a_{n\nu}(\Lambda_k) = \frac{1}{\nu!} \frac{d^\nu}{dz^\nu} \left\{ \frac{(z - \Lambda_k)^{p_k(n)}}{B_n(z)} \right\}_{z=\Lambda_k} \quad (\nu \geqslant 0,\ 1 \leqslant k \leqslant n). \qquad \ldots(1.4)$$

Theorem A_1 (Djrbashian)[6]—The systems $\{\tau_j(z)\}_1^n$ and $\{\Omega_{nk}(z)\}_1^n$ are biorthogonal on the real axes in the sense that

$$-\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \tau_j(\xi)\ \overline{\Omega_{nk}(\xi)}\ d\xi = \delta_{kj} \quad (1 \leqslant k, j \leqslant n). \qquad \ldots(1.5)$$

Note that, since Ω_{nk} is a rational function with poles at the points of sequence $\{\bar{\Lambda}_j\}_1^n$, it can be expanded by means of simple rational functions, and hence

$$\Omega_{nk}(z) = \sum_{j=1}^{n} c_{jk}\, \tau_j(z) \quad (1 \leqslant k \leqslant n). \qquad \ldots(1.6)$$

Hence, by (1.5) it is clear, that

$$c_{jk} = -\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Omega_{nk}(\xi)\ \overline{\Omega_{nj}(\xi)}\ d\xi \quad (1 \leqslant k, j \leqslant n). \qquad \ldots(1.7)$$

(b) Let us consider the following extremal problem : To find

$$N_{L^2(\mathbf{R})}\left(\{\tau_j\}_1^n ; k \right) = \max \{ |\, a_k \,| \} \quad (1 \leqslant k \leqslant n) \qquad \ldots(1.8)$$

where the maximum is taken on the set of generalized polynomials of the form

$$P(z) = \sum_{j=1}^{n} a_j\, \tau_j(z) \qquad \ldots(1.9)$$

with the norm

$$\|P\|_{L^2(\mathbf{R})} = \{ \int_{-\infty}^{+\infty} | P(\xi) |^2 \, d\xi \}^{1/2} \leqslant 1. \qquad \ldots(1.10)$$

Taking into account (1.5) and (1.6), from (0.4) we'll have

$$N_{L^2(\mathbf{R})} \left(\{\tau_j\}_1^n : k \right) = \left\| \frac{1}{2\pi i} \, \Omega_{nk} \right\|_{L^2(\mathbf{R})} \; (1 \leqslant k \leqslant n). \qquad \ldots(1.11)$$

In order to calculate these norms let us note, that from the identity $| B_n(\xi) | = 1$, $\xi \in \mathbf{R}$, and the definition (1.3) of the functions $\Omega_{nk}(z)$ the following equalities follow

$$\left\| \frac{1}{2\pi i} \, \Omega_{nk} \right\|_{L^2(\mathbf{R})} = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \Omega_{nk}(\xi) \, \overline{\Omega_{nk}(\xi)} \, d\xi$$

$$= \sum_{\nu, j=0}^{p_k(n)-s_k} \frac{i \, a_{n\nu}(\Lambda_k) \, \overline{a_{nj}(\Lambda_k)}}{2\pi \, [(s_k - 1)!]^2} \, \frac{1}{2\pi i} \int_{-\infty}^{+\infty}$$

$$\frac{d\xi}{(\xi - \Lambda_k)^{p_k(n)-s_k-\nu+1} \, (\xi - \overline{\Lambda}_k)^{p_k(n)-s_k-j+1}}.$$

Calculating the residue of integrand in the above integral at the point $z = \Lambda_k$ we get

$$\left\| \frac{1}{2\pi i} \, \Omega_{nk} \right\|_{L^2(\mathbf{R})} = \left\{ \sum_{\nu, j=0}^{p_k(n)-s_k} \frac{a_{n\nu}(\Lambda_k) \, \overline{a_{nj}(\Lambda_k)}}{2\pi \, [(s_k - 1)!]^2} \, \phi_{nk}(\nu; j) \right\}^{1/2} \qquad \ldots(1.12)$$

where for the values $0 \leqslant \nu, j \leqslant p_k(n) - s_k$

$$\phi_{nk}(\nu; j) = \frac{\Gamma(2p_k(n) - 2s_k - \nu - j + 1) \, i^{j-\nu}}{\Gamma(p_k(n) - s_k - \nu + 1) \, \Gamma(p_k(n) - s_k - j + 1) \, (2 \operatorname{Im} \Lambda_k)^{2p_k(n)-2s_k-\nu-j+1}} \qquad \ldots(1.13)$$

($\Gamma(z)$ is Euler's gamma-function).

Thus, from (1.11) and (1.12) we have

Theorem 1—If $1 \leqslant k \leqslant n$, then

$$N_{L^2(\mathbf{R})} \left(\{\tau_j\}_1^n ; k \right) = \left\{ \sum_{\nu, j=0}^{p_k(n)-s_k} \frac{a_{n\nu}(\Lambda_k) \, \overline{a_{nj}(\Lambda_k)}}{2\pi \, [(s_k - 1)!]^2} \, \phi_{nk}(\nu; j) \right\}^{1/2} \qquad \ldots(1.14)$$

moreover the external generalized polynomial is the rational function

$$P^*_{nk}(z) = - i \, \Omega_{nk}(z) / \|\Omega_{nk}\|_{L^2}(\mathbf{R}). \qquad \ldots(1.15)$$

In the particular case, when members of the sequence $\{\Lambda_j\}_1^n$, are different (so $s_k=p_k(n)=1$ $(1 \leqslant k \leqslant n)$) then our formulae become essentially simpler. In this case

$$\tau_j(z) = (z - \bar{\Lambda}_j)\ (1 \leqslant j \leqslant n)$$

and

$$\Omega_{nk}(z) = \frac{2i\,(\operatorname{Im}\Lambda_k)\,B_n(z)}{B_{nk}(z)\,(\Lambda_k)\,(z-\Lambda_k)}\ (1 \leqslant k \leqslant n) \qquad \ldots(1.3')$$

where

$$B_{nk}(z) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{z-\Lambda_j}{z-\bar{\Lambda}_j}\ (1 \leqslant k \leqslant n). \qquad \ldots(1.16)$$

Besides, in this case we'll have

$$N_{L^2(\mathbf{R})}\left(\{\tau_j\}_1^n;\, k\right) = \frac{1}{|\,B_{nk}(\Lambda_k)\,|}\sqrt{\frac{\operatorname{Im}\Lambda_k}{\pi}}\ (1 \leqslant k \leqslant n). \qquad \ldots(1.14')$$

Moreover the extremal generalized polynomial is the rational function

$$P_{nk}^*(z) = \sqrt{\frac{\operatorname{Im}\Lambda_k}{\pi}}\ \frac{|\,B_{nk}(\Lambda_k)\,|}{B_{nk}(\Lambda_k)}\ \frac{B_n(z)}{z-\Lambda_k}\ (1 \leqslant k \leqslant n). \qquad \ldots(1.15)$$

§ 2. The system $\{\exp(-\lambda_j t)\, t^{s_j-1}\}_1^n$ and $\{\omega_{nk}(t)\}_1^n$, are biorthogonal on the $(0, +\infty) = \mathbf{R}_+$.

(a) Let $\{\lambda_j\}_1^n$ $(\operatorname{Re}\lambda_j > 0)$ be any finite sequence of complex numbers from the right half-plane, and let S_k, $p_k(n)$ $(1 \leqslant k \leqslant n)$ be as defined earlier. In order to define the system $\{\omega_{nk}(t)\}_1^n$ which is biorthogonal to the system $\{\exp(-\lambda_j t)\, t^{s_j-1}\}_1^n$ set

$$\Lambda_j = i\,\bar{\lambda}_j\ (1 \leqslant j \leqslant n) \qquad \ldots(2.1)$$

and note that then $\operatorname{Im}\Lambda_j > 0$ $(1 \leqslant j \leqslant n)$ and for the sequences $\{\lambda_j\}_1^n$ and $\{\Lambda_j\}_1^n$ the corresponding multiplicities s_k and $p_k(n)$ $(1 \leqslant j \leqslant n)$ coincide. Further, by the Wiener-Paley theorem the transformation

$$F(z) = \frac{1}{\sqrt{2\pi}}\int_0^{+\infty} e^{itz} f(t)\, dt\ (\operatorname{Im} z > 0) \qquad \ldots(2.2)$$

defines a unitary operator U, which maps $L^2(\mathbf{R}_+)$ on the well-known space H_2 in the half-plane $\operatorname{Im} z > 0$. It is easy to check that

$$U(\exp(-\lambda_j t)\, t^{s_j-1}) = \frac{i^{s_j}}{\sqrt{2\pi}}\, \tau_j(z)\ (1 \leqslant j \leqslant n) \qquad \ldots(2.3)$$

where $\tau_j(z)$ are defined by (1.1). Hence, if we define the functions of the system $\{\omega_{nk}(t)\}_1^n$ from relations

$$U(\omega_{nk}(t)) = \frac{i^{s_k-1}}{\sqrt{2\pi}}\, \Omega_{nk}(z)\ (1 \leqslant k \leqslant n) \qquad \ldots(2.4)$$

[$\Omega_{nk}(z)$ are defined by (1.3)], then in view of unitarity of the operator U and owing to Theorem A_1 the systems of functions $\{\omega_{nk}(t)\}_1^n$ and $\{\varphi_j(t)\}_1^n$, where

$$\varphi_j(t) = \exp(-\lambda_j t)\, t^{S_j-1}\ (1 \leqslant j \leqslant n) \qquad \ldots(2.5)$$

will be biorthogonal on R_+. We will formulate this fact by a separate statement.

Theorem A_2—The systems of functions $\{\exp(-\lambda_j t)\, t^{S_j-1}\}_1^n$ and $\{\omega_{nk}(t)\}_1^n$ are biorthogonal on R_+ i.e.

$$\int_0^{+\infty} \exp(-\lambda_j t)\, t^{s_j-1}\, \overline{\omega_{nk}(t)}\, dt = \delta_{kj}\ (1 \leqslant k, j \leqslant n). \qquad \ldots(2.6)$$

Note that this statement actually had been established by Djrbashian[4,7], who considered the case of infinite system $\{\exp(-\lambda_j t) t^{s_j-1}\}_1^n$. To get Theorem A_2 it is enough to write $p_k(n)$ and $B_n(z)$ instead of p_k and $B(z)$ in the corresponding formulae.

In explicit form the functions of the system $\{\omega_{nk}(t)\}_1^n$ are defined as follows :

$$\frac{i^{S_k}}{2\pi i} \int_{-\infty}^{+\infty} e^{-t\xi t}\, \Omega_{nk}(\xi)\, d\xi = \begin{cases} \omega_{nk}(t), & t > 0 \\ 0, & t < 0, \end{cases} \qquad \ldots(2.7)$$

(integral converges in $L^2(R)$—norm). To see this it is sufficient to apply inversion formulas of the transform (2.2) to the identities (2.4) (cf. Djrbashian[2]).

Note at last, that from the equalities (1.6), (2.3) and (2.4) follow the representations

$$\omega_{nk}(t) = \sum_{j=1}^{n} \tilde{c}_{jk} \exp(-\lambda_j t)\, t^{s_j-1}\ (1 \leqslant k \leqslant n) \qquad \ldots(2.8)$$

where by (2.6) it is clear, that

$$\tilde{c}_{jk} = \int_0^{+\infty} \omega_{nk}(t)\, \overline{\omega_{nj}(t)}\, dt\ (1 \leqslant k \leqslant n). \qquad \ldots(2.9)$$

(b) Now let us consider the following extremal problem : To find

$$N_{L^2(\mathbf{R}_+)}\left(\{\varphi_j\}_1^n ; k\right) = \max\{\,|\,a_k\,|\,\}\ (1 \leqslant k \leqslant n) \qquad \ldots(2.10)$$

where the maximum is taken on the set of all possible generalized polynomials of the form

$$P(t) = \sum_{j=1}^{n} a_j \exp(-\lambda_j t)\, t^{s_j-1} \qquad \ldots(2.11)$$

with the norm

$$\|P\|_{L^2(\mathbf{R}_+)} = \{\int_0^{+\infty} |\,P(t)\,|^2\, dt\}^{1/2} \leqslant 1. \qquad \ldots(2.12)$$

Taking into account (2.6) and (2.8), and using (0.4) we have

$$N_{L^2(\mathbf{R}_+)}\left(\{\varphi_j\}_1^n ; k\right) = \|\omega_{nk}\|_{L^2(\mathbf{R}^+)}\ (1 \leqslant k \leqslant n). \qquad \ldots(2.13)$$

Moreover from the unitarity of the operator U and from the equality (2.4) it follows, that

$$\|\omega_{nk}\|_{L^2(\mathbf{R}_+)} = \left\|\frac{1}{\sqrt{2\pi}}\,\Omega_{nk}\right\|_{L^2(\mathbf{R})}\ (1 \leqslant k \leqslant n). \qquad \ldots(2.14)$$

From these equalities and from (1.12), (1.13) we get

Theorem 2—If $1 \leqslant k \leqslant n$, then

$$N_{L^2(\mathbf{R}_+)}\left(\{\varphi_j\}_1^n ; k\right) = \left\{\sum_{\nu,j=0}^{p_k(n)-s_j} \frac{a_{n\nu}(i\bar{\lambda}_k)\,\overline{a_{nj}(i\lambda_k)}}{\sqrt{2\pi}\,[(s_k-1)!]^2}\ \phi_{nk}(\nu; j)\right\}^{1/2} \qquad \ldots(2.15)$$

where $a_{n\nu}(i\bar{\lambda}_k)$ $(\nu \geqslant 0)$ are defined by the equalities (1.4) (with replacement $\{\Lambda_j\}_1^n$ by $\{i\bar{\lambda}_j\}_1^n$, while for $0 \leqslant \nu, j \leqslant p_k(n) - S_k$

$$\phi_{nk}(\nu; j) = \frac{\Gamma(2p_k(n) - 2s_k - \nu - j + 1)\, i^{j-\nu}}{\Gamma(p_k(n)-s_k-\nu+1)\,\Gamma(p_k(n)-s_k-j+1)\,(2\mathrm{Re}\lambda_k)^{2p_k(n)-2s_k-\nu-j+1}}. \qquad \ldots(2.16)$$

The extremal generalized polynomial is the fnnction

$$P^*_{nk}(t) = \omega_{nk}(t)/\|\omega_{nk}\|_{L^2(\mathbf{R}_+)}\,. \qquad \ldots(2.17)$$

(c) In the particular case, when members of the sequence $\{\lambda_j\}_1^n$ are pair-wise different (then $s_k = p_k(n) = 1$ $(1 \leqslant k \leqslant n)$), then the system (2.5) becomes the

system of functions $\{e^{-i\lambda_j t}\}_1^n$. In this case

$$\frac{i \operatorname{Re} \lambda_k}{\pi B_{nk}(i\bar{\lambda}_k)} \int_{-\infty}^{+\infty} e^{-i\xi t} \frac{B_n(\xi)}{\xi - i\bar{\lambda}_k} d\xi = \begin{cases} \omega_{nk}(t), & t > 0, \\ 0, & t < 0, \end{cases} \qquad \ldots(2.7')$$

where

$$B_n(z) = \prod_{j=1}^{n} \frac{z - i\bar{\lambda}_j}{z + i\lambda_j}, \quad B_{nk}(z) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{z - i\bar{\lambda}_j}{z + i\lambda_j} \quad (1 \leqslant k \leqslant n) \qquad (2.18)$$

and the integral converges in $L^2(\mathbf{R})$—norm. Besides,

$$N_{L^2(\mathbf{R}_+)}\left(\{\varphi_j\}_1^n ; k \right) = \sqrt{2 \operatorname{Re} \lambda_k} \; / \; | B_{nk}(i\bar{\lambda}_k) | \; (1 \leqslant k \leqslant n). \qquad \ldots(2.15')$$

Moreover, the extremal generalized polynomial $P_{nk}^*(t)$ is defined by the equation

$$\frac{i\sqrt{\operatorname{Re} \lambda_k} \; | B_{nk}(i\bar{\lambda}_k) |}{\pi \sqrt{2} \, B_{nk}(i\bar{\lambda}_k)} \int_{-\infty}^{+\infty} e^{-i\xi t} \frac{B_n(\xi)}{\xi - i\bar{\lambda}_k} d\xi = \begin{cases} P_{nk}^*(t), & t > 0, \\ 0, & t < 0, \end{cases} \qquad \ldots(2.17)$$

(the integral converges in the $L^2(\mathbf{R})$—norm).

In the special case, when $\lambda_1, \ldots, \lambda_n$ are real > 0 and different, the functions $\omega_{nk}(t)$ and $P_{nk}^*(t)$ were built by Schwartz[11] and he also established the equalities (2.15').

§ 3. The systems $\{x^{\mu_j} (\log x)^{S_j - 1}\}_1^n$ and $\left\{\omega_{nk}^*(x)\right\}_1^n$, are biorthogonal on the segment $[0, 1] = I$.

(a) Let $\{\mu_j\}_1^n$ ($\operatorname{Re} \mu_j > -1/2$) be any finite sequence of complex numbers, and S_k and $p_k(n)$ $(1 \leqslant k \leqslant n)$ are defined as be before. In order to define the system $\left\{\omega_{nk}^*(x)\right\}_1^n$, which is biorthogonal to $\{x^{\mu_j} (\log x)^{S_j - 1}\}_1^n$ we set

$$\lambda_j = \mu_j + \frac{1}{2} \quad (1 \leqslant j \leqslant n) \qquad \ldots(3.1)$$

and note, that $\operatorname{Re} \lambda_j > 0$ $(1 \leqslant j \leqslant n)$, moreover, for the sequences $\{\mu_j\}_1^n$ and $\{\lambda_j\}_1^n$, the corresponding multiplies S_k and $p_k(n)$ $(1 \leqslant k \leqslant n)$ coincide.

Further, it is easy to check, that the transforma tion

$$F(x) \to f(t) = e^{-t/2} F(e^{-t}) \ (x \in I, t \in \mathbf{R}_+) \qquad \ldots(3.2)$$

defines a unitary operator $V: L^2(I) \to L^2(\mathbf{R}_+)$, and

$$V(x^{\rho_j} (\log x)^{s_j-1}) = (-1)^{s_j-1} \exp(-\lambda_j t) t^{s_j-1} (1 \leqslant j \leqslant n). \qquad \ldots(3.3)$$

Hence, if the functions of the system $\left\{\omega_{nk}^*(x)\right\}_1^n$ are defined by the equations

$$V\left(\omega_{nk}^*(x)\right) = (-1)^{s_k-1} \omega_{nk}(t) \ (1 \leqslant k \leqslant n) \qquad \ldots(3.4)$$

then in view of the unitarity of operator V and by the Theorem A_2 the systems of functions $\left\{\omega_{nk}^*(x)\right\}_1^n$ and $\left\{\varphi_j^*(x)\right\}_1^n$, where

$$\varphi_j^*(x) = x^{\rho_j} (\log x)^{s_j-1} \ (1 \leqslant j \leqslant n) \qquad \ldots(3.5)$$

and biorthogonal on the segment $I = [0, 1]$.

Theorem A_3—The systems of functions $\left\{\varphi_j^*(x)\right\}_1^n$ and $\left\{\omega_{nk}^*(x)\right\}_1^n$ are biorthogonal on the segment [0, 1], i.e.

$$\int_0^1 x^{\rho_j} (\log x)^{s_j-1} \overline{\omega^*(x)} \, dx = \delta_{kj} \ (1 \leqslant k, j \leqslant n). \qquad \ldots(3.6)$$

From relations (3.2), (3.4) and (2.7) it follows easily that

$$\frac{i^{1-s_k}}{2\pi} \int_{-\infty}^{+\infty} \Omega_{nk}(\xi) x^{i\xi+1/2} \, d\xi = \begin{cases} \omega_{nk}^*(x), \ x \in (0, 1) \\ 0, \ x \in (1, +\infty), \end{cases} \qquad \ldots(3.7)$$

(the integral converges in $L^2(\mathbf{R}_+)$-norm), and to define the functions $\Omega_{nk}(z)$ it is necessary to set $\Lambda_j = i(\mu_j + 1/2)$ in the formulas (1.2), (1.3) and (1.4) (see (2.1) and (3.1)).

It is easy to find, that from (2.8), (3.3) and (3.4) it follows the representations

$$\omega_{nk}^*(x) = \sum_{j=1}^{n} c_{jk}^* x^{\rho_j} (\log x)^{s_j-1} \ (1 \leqslant k \leqslant n) \qquad \ldots(3.8)$$

moreover by (3.6) it is clear, that

$$c_{jk}^* = \int_0^1 \omega_{nk}^*(x) \overline{\omega_{nj}^*(x)} \, dx \ (1 \leqslant k, j \leqslant k). \qquad \ldots(3.9)$$

Note that, the systems of the functions (3.5) had been written in a less compact form by Gelfond[8] in connection with a generalization of the known polynomials of Bernstein[1] and Haussdorff[9]. The above mentioned was suggested by Dzrbashian[7].

(b) Let us consider the following extremal problem : To find

$$N_{L^2(I)}\left(\left\{\varphi_j^*\right\}_1^n;\ k\right) = \max\{\,|\,a_k\,|\,\}\ (1 \leqslant k \leqslant n) \qquad \ldots(3.10)$$

where the maximum is taken on the set of all possible generalized polynomials of the form

$$P(x) = \sum_{j=1}^{n} a_j\, x^{\mu_j}\,(\log x)^{\varkappa_j - 1} \qquad \ldots(3.11)$$

with the norm

$$\|P\|_{L^2(I)} = \{\int_0^1 |\,P(x)\,|^{\,2}\,dx\}^{1/2} \leqslant 1. \qquad \ldots(3.12)$$

Using (3.6), (3.8) and (0.4) we get

$$N_{L^2(I)}\left(\left\{\varphi_j^*\right\}_1^n;\ k\right) = \|\omega_{nk}^*\|_{L^2(I)}\ (1 \leqslant k \leqslant n) \qquad \ldots(3.13)$$

moreover from the unitarity of operator V and (3.4) it follows that

$$\|\,\omega_{nk}^*\,\|_{L^2(I)} = \|\omega_{nk}\|_{L^2(\mathbf{R}_+)}\ (1 \leqslant k \leqslant n). \qquad \ldots(3.14)$$

From (3.13) and (3.14) and according to the equalities (2.13), (2.15), and (2.16) we obtain

Theorem 3—If $1 \leqslant k \leqslant n$, then

$$N_{L^2(I)}\left(\left\{\varphi_j^*\right\}_1^n;k\right)=\left\{\sum_{\nu,j=0}^{p_k(n)-s_k} \frac{a_{n\nu}\left(i\bar{\mu}_k+\frac{1}{2}\right)\overline{a_{n\nu}\left(i\bar{\mu}_j+\frac{1}{2}\right)}}{\sqrt{2\pi}\,[(s_k-1)\,!]^2}\,\phi_{nk}(\nu;j)\right\}^{1/2}, \qquad \ldots(3.15)$$

where $a_{n\nu}(i\bar{\mu}_j + i/2)$ $(\nu \geqslant 0)$ are defined by (1.4) (with substitution of $\{\Lambda_j\}_1^n$ by $\{i\bar{\mu}_j + i/2\}_1^n$), while for

$$\phi_{nk}(\nu,j) = \frac{\Gamma(2p_k(n) - 2s_k - \nu - j + 1)\, i^{j-\nu}}{\Gamma(p_k(n)-s_k-\nu+1)\Gamma(p_k(n)-s_k-j+1)(2\mathrm{Re}\,\mu_k+1)^{2p_k(n)-2s_k-\nu-j+1}}. \qquad \ldots(3.16)$$

The extremal generalized polynomial is the function

$$P_{nk}^{*}(x) = \omega_{nk}^{*}(x)/\|\omega_{nk}^{*}\|_{L^2(I)}.$$

(c) In the particular case, when terms of the sequence $\{\mu_j\}_1^n$ are pair-wise different (so $S_k = p_k(n) = 1$ $(1 \leqslant k \leqslant n)$), then the system (3.5) becomes the system of Müntz $\{x^{\mu_j}\}_1^n$. In this case

$$\frac{i(\operatorname{Re}\mu_k + 1/2)}{\pi B_{nk}(i\bar{\mu}_k - i/2)} \int_{-\infty}^{+\infty} \frac{B_n(\xi)}{\xi - i\bar{\mu}_k - i/2}\, x^{i\xi+1/2}\, d\xi = \begin{cases} \omega_{nk}^{*}(x), & x \in (0, 1), \\ 0, & x \in (1, +\infty), \end{cases} \qquad \ldots(3.17)$$

where

$$B_n(z) = \prod_{j=1}^{n} \frac{z - j\bar{\mu}_j - i/2}{z + i\mu_j + i/2}, \quad B_{nk}(z) = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{z - i\bar{\mu}_j - i/2}{z + i\mu_j + i/2} \quad (1 \leqslant k \leqslant n), \qquad \ldots(3.18)$$

and the integral converges in $L^2(\mathbf{R}_+)$-norm.

Besides,

$$N_{L^2(I)}\left(\{\varphi_j^{*}\}_1^n; k\right) = \frac{\sqrt{2\operatorname{Re}\mu_k + 1}}{|B_{nk}(i\bar{\mu}_k + i/2)|} \quad (1 \leqslant k \leqslant n) \qquad \ldots(3.15')$$

and the extremal generalized polynomial $P_{nk}^{*}(x)$ is defined from the equality

$$\frac{i\sqrt{2\operatorname{Re}\mu_k + 1}\,|B_{nk}(i\bar{\mu}_k + i/2)|}{2\pi B_{nk}(i\bar{\mu}_k + i/2} \int_{-\infty}^{+\infty} \frac{B_n(\xi)}{\xi - i\bar{\mu}_k - i/2}\, d\xi^{i\xi+1/2}$$

$$d\xi = \begin{cases} P_{nk}^{*}(x), & x \in (0, 1) \\ 0, & x \in (1, +\infty) \end{cases} \qquad \ldots(3.17')$$

(the integral converges in $L^2(\mathbf{R}_+)$-norm).

In the special case, when $\mu_1, \ldots, \mu_n$ are reals $> -1/2$ and different, then the functions $\omega_{nk}^{*}(x)$ and $P_{nk}^{*}(x)$ were built by Schwartz[11] and the equalities (3.15′) were established by him.

§ 4. The systems $\{\tau_j(\zeta)\}_1^n$ and $\{\Omega_{nk}(\zeta)\}_1^n$, are biorthogonal on the unit circle $\{\zeta : |\zeta| = 1\} = \partial \mathbb{D}$.

(a) Let $\{\alpha_j\}_1^n$ $(|\alpha_j| < 1)$ be any finite sequence of complex numbers from the unit disc $\mathbb{D} = \{\zeta : |\zeta| < 1\}$, and s_k and $p_k(n)$ $(1 \leqslant k \leqslant n)$ are defined as before. In this case the system of rational functions $\{\tau_j(\zeta)\}_1^n$ is defined as follows:

$$\tau_j(\zeta) = \frac{(s_j - 1)!\,\zeta^{s_j-1}}{(1 - \bar{\alpha}_j \zeta)^{s_j}} \quad (1 \leqslant j \leqslant n) \qquad \ldots(4.1)$$

(see Djrbashian[5]). Let us denote by

$$B_n(\zeta) = \prod_{j=1}^{n} \frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j \zeta} \qquad \ldots(4.2)$$

the finite Blaschke product for the unit disc and, as in Djrbashian[5], we consider a system of functions

$$\Omega_{nk}(\zeta) = \frac{B_n(\zeta)}{(s_k - 1)!} \sum_{\nu=0}^{p_k(n)-s_k} \frac{a_{n\nu}(\alpha_k)}{(\zeta - \alpha_k)^{p_k(n)-s_k-\nu+1}} \quad (1 \leqslant k \leqslant n) \qquad \ldots(4.3)$$

where

$$a_{n\nu}(\alpha_k) = \frac{1}{\nu!} \frac{d^\nu}{d\zeta^\nu} \left\{ \frac{(\zeta - \alpha_k)^{p_k(n)}}{B_n(\zeta)} \right\}_{\zeta=\alpha_k} \quad (\nu \geqslant 0,\ 1 \leqslant k \leqslant n). \qquad \ldots(4.4)$$

Theorem A_4 (Djrbashian[5])—The systems of functions $\{\tau_j(\zeta)\}_1^n$ and $\{\Omega_{nk}(\zeta)\}_1^n$ are biorthogonal on the circle $\partial \mathbb{D}$ in the sense

$$\frac{1}{2\pi} \int_{|\zeta|=1} \tau_j(\zeta)\, \overline{\Omega_{nk}(\zeta)}\, |d\zeta| = \delta_{kj} \quad (1 \leqslant k, j \leqslant n). \qquad \ldots(4.5)$$

Since $\Omega_{nk}(\zeta)$ is a rational function with poles in the points of sequence $\{1/\bar{\alpha}\}_1^n$ then it may be expanded as a sum af the simple rational fractions :

$$\Omega_{nk}(\zeta) = \sum_{j=1}^{n} c_{jk}\, \tau_j(\zeta) \quad (1 \leqslant k \leqslant n). \qquad \ldots(4.6)$$

According to (4.5) we'll have :

$$c_{jk} = \frac{1}{2\pi} \int_{|\zeta|=1} \Omega_{nk}(\zeta)\, \overline{\Omega_{nj}(\zeta)}\, |d\zeta| \quad (1 \leqslant k, j \leqslant n). \qquad \ldots(4.7)$$

(b) Let us consider the following extremal problem : To find

$$N_{L^2(\partial\,\mathbb{D})}\left(\{\tau_j\}_1^n ; k\right) = \max\{\,|\,a_k\,|\,\}\,(1 \leqslant k \leqslant n) \qquad \ldots(4.8)$$

where the maximum is taken on the set of generalized polynomials of the form

$$P(\zeta) = \sum_{j=1}^{n} a_j\,\tau_j(\zeta) \qquad \ldots(4.9)$$

with the norm

$$\|P\|_{L^2(\partial\,\mathbb{D})} = \{\int_{|\zeta|=1} |\,P(\zeta)\,|^2\,|\,d\zeta\,|\}^{1/2} \leqslant 1. \qquad \ldots(4.10)$$

Taking into accout (4.5) and (4.6), and by (0.4) we'll have:

$$N_{L^2(\partial\,\mathbb{D})}\left(\{\tau_j\}_1^n ; k\right) = \left\|\frac{1}{2\pi}\,\Omega_{nk}\right\|_{L^2(\partial\,\mathbb{D})}\,(1 \leqslant k \leqslant n). \qquad \ldots(4.11)$$

Since $|\,B_n(\zeta)\,| = 1$, $\zeta \in \partial\,\mathbb{I}$, then in view of (4.3) the following equalities are true :

$$\left\|\frac{1}{2\pi}\,\Omega_{nk}\right\|^2_{L^2(\partial\,\mathbb{D})} = \frac{1}{(2\pi)^2}\int_{|\zeta|=1} \Omega_{nk}(\zeta)\,\overline{\Omega_{nk}(\zeta)}\,|\,d\zeta\,|$$

$$= \sum_{\nu,j=0}^{p_k(n)-s_k} \frac{a_{n\nu}(\alpha_k)\,\overline{a_{nj}(\alpha_k)}}{2\pi\,[(s_k-1)\,!]^2}$$

$$\times\ \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{\zeta^{p_k(n)-s_k-j}\,d\zeta}{(\zeta-\alpha_k)^{p_k(n)-s_k-\nu+1}\,(1-\bar{\alpha}_k\,\zeta)^{p_k(n)-s_k-j+1}}\,.$$

Calculating residue of the above integrand at the point $\zeta = \alpha_k$ we get

$$\left\|\frac{1}{2\pi}\,\Omega_{kn}\right\|_{L^2(\partial\,\mathbb{D})} = \left\{\sum_{\nu,j=1}^{p_k(n)-s_k} \frac{a_{n\nu}(\alpha_k)\,\overline{a_{nj}(\alpha_k)}}{2\pi\,[(s_k-1)\,!]^2}\,\Psi_{nk}(\nu;j)\right\}^{1/2} \qquad \ldots(4.12)$$

where for the values $0 \leqslant \nu, j \leqslant p_k(n) - s_k$

$$\Psi_{nk}(\nu;j) = \frac{1}{(p_k n) - s_k - \nu)!}\,\frac{d^{p_k(n)-s_k-\nu}}{d\zeta^{p(n)-s_k-\nu}}\left\{\frac{\zeta^{p_k(n)-s_k-j}}{(1-\bar{\alpha}_k\,\zeta)^{p_k(n)-s_k-j+1}}\right\}_{\zeta=\alpha_k}. \qquad \ldots(4.13)$$

Theorem 4—If $1 \leqslant k \leqslant n$, then

$$N_{L^2(\partial\,\mathbb{D})}\left(\{\tau_j\}_1^n ; k\right) = \left\{\sum_{\nu,j=0}^{p_k(n)-s_k} \frac{a_{n\nu}(\alpha_k)\,\overline{a_{nj}(\alpha_k)}}{2\pi\,[(s_k-1)\,!]^2}\,\Psi_{nk}(\nu;j)\right\}^{1/2} \qquad \ldots(4.14)$$

moreover the extremal generalized polynomial is the rational function

$$P^*_{nk}(\zeta) = \Omega_{nk}(\zeta)/\|\Omega_{nk}\|_{L^2(\partial\,\mathbb{D})}. \qquad \ldots(4.15)$$

(c) Now let $\alpha_1,,\ldots,\alpha_n$ be different (then $S_k = p_k(n)$ $(1 \leqslant k \leqslant n)$). Then $\tau_j(\zeta) = (1 - \bar{\alpha}_j\zeta)^{-1}$ and

$$\Omega_{nk}(\zeta)/= \frac{1 - |\alpha_k|^2}{B_{nk}(\alpha_k)} B_n(\zeta)/(\zeta - \alpha_k)\ (1 \leqslant k \leqslant n), \qquad \ldots(4.3')$$

where

$$B_{nk}(\zeta) = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j\zeta}\ (1 \leqslant k \leqslant n). \qquad \ldots(4.16)$$

In this case we will have :

$$N_{L^2(\partial\,\mathbb{D})}\left(\{\tau_j\}_1^n ; k\right) = |B_{nk}(\alpha_k)|^{-1}\ (1 \leqslant k \leqslant n) \qquad \ldots(4.14')$$

and the extremal generalized polynomial is the rational function

$$P^*_{nk}(\zeta) = (1 - |\alpha_k|^2)\ \frac{B_n(\zeta)}{(\zeta - \alpha_k)}\ (1 \leqslant k \leqslant n). \qquad \ldots(4.15')$$

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