Indian J. pure appl. Math., 17 (7) : 919-924, July 1986

ON GENERALISED ADJOINT ABELIAN OPERATORS ON BANACH SPACES

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(*Received* 21 *March* 1985; *after revision* 7 *August* 1985)

In this paper Riesz representation theorem is proved in a generalised semi-inner-product space. The notion of a generalised adjoint abelian operator T on a Banach space X is introduced and shown that the null space $N(T)$ is complemented; in fact $X = N(T) \oplus \overline{R(T)}$. Finally, the countability of the point spectrum of an adjoint abelian operator is obtained.

A generalised semi-inner-product (generalisation of semi-inner-product of Lumer[7]) on a complex linear space X is a complex function $[\ldots]$ on $X \times X$, which is linear in the first argument only, strictly positive, and satisfies the Holder's inequality that for all x, y in X,

$$|[x, y]| \leqslant [x, x]^{1/p} [y, y]^{(p-1)/p}$$

where p can be any number $\geqslant 1$. A linear space equipped with a generalised semi-inner-product (g.s.i.p) is called a generalised semi-inner-product space (g.s.i.p. space). A g.s.i.p space is a normed linear space whose norm is given by $\|x\| = [x, x]^{1/p}$. A homogeneous g.s.i.p space is one whose g.s.i.p. $[.,.]$ satisfies the additional condition that $[x, \alpha y] = \bar{\alpha} |\alpha|^{p-2} [x, y]$ for all complex α. For more details on g.s.i.p. we refer to Nath[10].

For $p = 2$, g.s.i.p is an s.i.p. of Lumer[7].

For each x in a normed linear space X, and for all p with $1 < p < \infty$, there is a functional $x^* \in X^*$ such that $\|x^*\| = \|x\|^{p-1}$ and $x^*(x) = \|x\|^p$ (Nath[10], Lemma II). Let ϕ associate each x in X to exactly one such x^* in X^* and αx to $\bar{\alpha} |\alpha|^{p-2} x^*$ (for all complex α). We call ϕ a generalised support mapping. A g.s.i.p which produces the norm of X is termed either as a consistent or compatible g.s.i.p. Each generalised support mapping defines a consistent homogeneous g.s.i.p on X if we set $(\phi(x))(y) = [y, x]$ and hence every normed linear space can be considered as a homogeneous g.s.i.p. space. In the rest of the paper all operators are bounded linear operators on complex Banach spaces, and all g.s.i.p's are compatible ones.

We now introduce:

Definition 1—An operator A on X is said to be a generalised adjoint abelian operator (g.A.A.o) if there is a generalised support mapping ϕ such that $A^* \phi = \phi A$, where A^* is the adjoint of A.

We note that the following conditions are equivalent for an operator A on a Banach space X.

(i) $A^*\phi = \phi A$

(ii) $(Ax)^* = A^*x^*$

(iii) $[Ax, y] = [x, Ay]$ for all x, y in X, where $[.\ ,\ .]$ is the g.s.i.p on X produced by ϕ.

For $p = 2$, a g.A.A.o is an adjoint abelian operator of Stampfli[12].

Definition 2—A Banach space X is said to be weakly locally uniformly convex (respectively, locally uniformly convex) whenever given $(x_n) \subset S(X)$ and $x_0 \in S(X)$ for which

$$\frac{\|x_n + x_0\|}{2} \to 1$$

then $x_n \to x_0$ weakly (respectively, in norm) (Diestel[1], p. 31).

A normed linear space is said to be strictly convex if each point of the unit sphere is an extreme point of the unit ball.

Lemma 3—If x and y are nonzero vectors in a strictly convex g.s.i.p space X with $[x, y] = \|x\| \|y\|^{p-1}$ $(1 < p < \infty)$ then $y = \alpha x$ for some $\alpha > 0$.

The proof is simple and hence omitted.

Lemma 4—Every reflexive Banach space can be given an equivalent norm such that the space is locally uniformly convex, and hence strictly convex.

This follows from Theorem 3 (p. 31) and Corollary 3 (p.167) of Diestel[1].

A vector x in a normed linear space X is said to be normal (orthogonal w.r.t. a g.s.i.p $[.\ ,\ .]$ on X) to $y \in X$ if $\|x + \mu y\| \geq \|x\|$ for all complex μ ($[y, x] = 0$). M is said to be normal (orthogonal) to N if each $x \in M$ is normal (orthogonal) to all $y \in N$, where M and N are subsets of X.

Lemma 5—Let M and N be subspaces of a normed linear space X, then M is normal to N if and only if there is a homogeneous g.s.i.p $[.,.]$ on X with respect to which M is orthogonal to N.

The proof is omitted since it can be proved by suitably modifying that of Faulkner[2] (Theorem 1).

Suppose $f \in X^*$ with $|f(x)| = \|f\| \|x\|$ for some $x \in X$ then $\|f\| \|x + n\| \geq |f(x + n)| = |f(x)| = \|f\| \|x\|$; shows that x is normal to $N(f)$. Mazur[8] shows that in a reflexive Banach space, for any $f \in X^*$, there is an $x \in X$ such that $|f(x)| = \|f\| \|x\|$.

In view of this we prove the Riesz representation theorem.

Theorem 6—If X is a reflexive space, then for each $f \in X^*$, there is a g.s.i.p $[.\ ,\ .]$ and an element $y \in X$ such that $f(x) = [x, y]$ for all x in X.

PROOF : If $N(f) = X$, we can choose any g.s.i.p with $y = 0$. Suppose $N(f) \neq X$. Since X is reflexive, by the preceeding remark there is an x_0 in X such that x_0 is normal to $N(f)$. So, M, the span of x_0, is normal to $N(f)$. By Lemma 5, there is a homogeneous g.s.i.p $[.,.]$ on X such that M is orthogonal to $N(f)$: Let $w = f(x)\, x_0 - f(x_0)\, x$, $x \in X$. Since $w \in N(f)$

$$0 = [w, x_0] = f(x)\, \|x_0\|^p - f(x_0)\, [x, x_0] \quad (1 < p < \infty) \qquad \ldots(1)$$

since the g.s.i.p $[.\ ,\ .]$ is homogeneous, we can write (1) as

$$f(x) = [x, \alpha x_0] = [x, y]$$

where $y = \alpha x_0$ and α is such that $\alpha f(x_0) = \|\alpha x_0\|^p$.

Theorem 7—In the case that X is also strictly convex the vector y in Theorem 6 is unique.

Proof follows by Lemma 3.

Remark 8: By Lemma 4, any reflexive space can be given an equivalent strictly convex norm. Now by choosing a g.s.i.p consistent with this equivalent norm, we may assume that the representing element in Theorem 6 is unique and hence the result of Milicie[9]. (Theorem 4) is contained in Theorem 6. In particular, for $p = 2$, the representation theorem of Giles[3], Husain and Malaviya[5], Unni and Puttamadaiah[14], and Faulkner[2], proved under different conditions, is a consequence of Theorem 6.

Corollary 9— If X is a reflexive space, then there is a nonzero vector orthogonal to every proper closed subspace M of X.

PROOF : If $y \notin M$, then there is an $f \in X^*$ such that $f(M) = 0$ and $f(y) \neq 0$. By Theorem 6, there is a g.s.i.p $[.\ ,\ .]$ on X and a vector z in X such that $f(x) = [x, z]$ for all x in X. Since $f(M) = 0$, z is orthogonal to M.

If A is orthogonal to B with respect to a g.s.i.p then clearly A is orthogonal to $\bar{B}$.

Theorem 10—Let T be a g.A.A.o on a Banach space X, then $N(T)$ and $\overline{R(T)}$ are orthognal to each other.

PROOF : Since T is a g.A.A.o, there is a g.s.i.p $[.\ ,\ .]$ on X such that $[Tx, y] = [x, Ty]$ for all x, y in X. If $x \in N(T)$ and $y \in R(T)$, then

$$[y, x] = [Tz, x] = [z, Tx] = 0$$

so $N(T)$ is orthogonal to $R(T)$. Similarly $R(T)$ is orthogonal to $N(T)$.

Let T be a linear operator on a vector space X. If there exists a positive integer p such that $N(T^p) = N(T^{p+1})$, the smallest such integer is called the ascent of T. If there is some positive integer q such that $R(T^q) = R(T^{q+1})$, the smallest such integer is called the descent of T.

Corollary 11—If T is a g.A.A.o on a Banach space X; then the ascent of T is $\leqslant 1$.

PROOF : Since by Theorem 10, $N(T)$ is orthogonal to $R(T)$, it follows that $N(T^2) \subset N(T)$. Since $N(T) \subset N(T^2)$ always, the ascent of T is $\leqslant 1$.

Theorem 12—If T is a g.A.A.o on a reflexive Banach space X, then $X = N(T) \oplus \overline{R(T)}$.

PROOF : First we show that $N(T) + R(T)$ is dense in X. Since T is a g.A.A.o, there is a g.s.i.p $[.\,,.]$ on X such that $[Tx, y] = [x, Ty]$ for all x and y in X. Suppose $N(T) + R(T)$ is not dense in X, then by Corollary 9, there is a non-zero $z \in X$ othogonal to $N(T) + R(T)$; so $[x, z] = 0 = [y, z]$ for an $x \in N(T)$ and $y \in R(T)$. If $y = Tp$ for $p \in X$ then $[Tp, z] = [p, Tz] = 0$ for all $p \in X$ implies $Tz = 0$, and hence $z \in N(T)$. Since z is orthogonal to $N(T)$, $z = 0$ which is not true. We now show that $N(T) + \overline{R(T)}$ is closed in X. By Theorem 10, $N(T)$ and $\overline{R(T)}$ are orthogonal to each other. Let (z_n) be a sequence in $N(T) + \overline{R(T)}$ such that $z_n \to z$. Let $z_n = x_n + y_n$, $x_n \in N(T)$ and $y_n \in \overline{R(T)}$ then

$$\begin{aligned} \|z_n - z_m\| \, \|y_n - z_m\|^{p-1} &\geqslant |\, [y_n, -z_m\, y_n - y_m]\,| \\ &= |\, [x_n - x_m, y_n - y_m] + [y_n - y_m, y_n - y_m]\,| \\ &= \|y_n - y_m\|^p. \end{aligned}$$

Hence (y_n) is a cauchy sequence and so $y_n \to y \in \overline{R(T)}$. Since $x_n = z_n - y_n \to z - y$, if $z - y = x$, then $x \in N(T)$. Thus $z_n \to x + y = z \in N(T) + \overline{R(T)}$, and so it is closed. Since $N(T) \cap \overline{R(T)} = \{0\}$, $X = N(T) \oplus \overline{R(T)}$.

Corollary 13—If T is a g.A.A.o on a reflexive Banach space X, then T is one-one if and only if $R(T)$ is dense in X.

Corollary 14—If T is a g.A.A.o on a reflexive Banach space, then $\overline{R(T)} = \overline{R(T^2)}$

PROOF : By Theorem 12, $X = N(T) \oplus \overline{R(T)}$. Since T^2 is a g.A.A.o whenever T is so, again by Theorem 12, $X = N(T^2) \oplus \overline{R(T^2)}$. By Corollary 11, $N(T) = N(T^2)$, and so $\overline{R(T)} = \overline{R(T^2)}$.

Corollary 15—If T is a g.A.A.o on a reflexive Banach space X and its descent is finite, then $R(T)$ is closed.

PROOF : Since T is a g.A.A.o, by Theorem 12, $X = N(T) \oplus \overline{R(T)}$, and the ascent $\alpha(T)$ of T is $\leqslant 1$ (Corollary 11). By hypothesis the descent $\partial(T)$ of T is finite and hence $\alpha(T) = \partial(T) = 1$ (Taylor[13], Theorem 5.41-E, p. 273) and hence $X = N(T) \oplus R(T)$ (Taylor[13], Theorem 5.41-G, p. 273). Thus $R(T) = \overline{R(T)}$.

A continuous g.s.i.p space X is a g.s.i.p space where the g.s.i.p has the additional property :

For every $x, y \in X$, $\operatorname{Re}[y, x + \mu y] \to \operatorname{Re}[y, x]$ for all real $\mu \to 0$. The norm of a normed linear space is Gateaux (or weakly) differentiable if for all $x, y \in X$ and real μ.

$$\lim_{\mu \to 0} \frac{\|x + \mu y\| - \|x\|}{\mu} \text{ exists.}$$

A normed linear space X is said to be smooth if for each unit vector $x \in X$, the set $D(x) = \{f \in X^* : \|f\| = f(x) = 1\}$ contains only one point.

We note that if a normed linear space X is smooth then it has a unique compatible g.s.i.p.

We note that the following conditions on X are equivalent :

(i) continuity of a g.s.i.p

(ii) Gateaux differentiability of the norm

(iii) smoothness

(Giles[3], Theorem 2; James[6], p. 274; and Giles[4], Theorem 1 and Lemma 3).

Theorem 16—If T is an adjoint abelian operator on a smooth separable Banach space X then the point spectrum $\sigma_p(T)$ of T is countable.

PROOF : Since X is smooth there is a unique compatible s.i.p. $[.\,,.]$ on X. Since T is adjoint abelian, by (Stampfli[12], Lemma 4, p. 506), $[x_\alpha, x_\mu] = 0 = [\mu, x_\alpha]$ for $\alpha \neq \mu$, where $x_\lambda \in N(T - \lambda I)$ with $\|x_\lambda\| = 1$. By the above note, the s.i.p $[.\,,.]$ is continuous. Hence by (Giles[3], Theorem 2).

$$\|x_\alpha - x_\mu\| \geq 1 \quad (\alpha \neq \mu)$$

So, the members of the family of open balls in X with centre x_α and radius $1/2$ are disjoint. Since X is separable this family must be countable and hence the set $\{x_\alpha : \alpha \in \sigma_p(T)\}$ is countable; and therefore $\sigma_p(T)$ is countable.

Remark : It would be interesting to know whether Theorem 16 is true for a g.A.A.o.

ACKNOWLEDGEMENT

The authors express their sincere thanks to the referee for his valuable suggestions and comments.

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