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ON EQUALITY OF SPACES $LH^{\beta}_{a,\mu,\alpha}$ AND $LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu}$

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It is proved that the intersection of the spaces $LH_{a,\mu,\alpha}$ and $LH^{\beta}_{a\mu}$ is the space $LH^{\beta}_{a,\mu,\alpha}$ where $LH_{a,\mu,\alpha}$, $LH^{\beta}_{a\mu}$ and $LH^{\beta}_{a,\mu,\alpha}$ are spaces of type $LH_{a,\mu}$.

INTRODUCTION

The theory of generalized integral transforms (distributional intgeral transforms) has its origin in the work of Schwartz[1] who first extended Fourier transform to generalized functions. Gelfand and Shilov[2] introduced spaces of type S and used them to solve Cauchy problem. Lee[3] introduced spaces of type H_μ and used them for the study of Dirichlet problem in cylindrical coordinates.

Motivated from the work of Gelfand and Shilov and Lee, the author (Tiwari[4,5]) introduced spaces of type $LH_{a,\mu}$ and discussed its several properties. In Tiwari[6] these spaces are used to solve the equation of propagation of heat in cylindrical coordinates imposing the generalized boundary conditions. Several topological properties of these spaces are proved in Tiwari[7].

We denote by Ω_1 the open set $(R_+) \times (R_+)$. $E(\Omega_1)$ denotes the class of infinitely differentiable functions defined on Ω_1. The author defined the spaces of type $LH_{a,\mu}$ as follows :

The space $LH_{a,\mu,\alpha}$: It is defined as (see Tiwari[4], p. 1533 or Tiwari[8], p.49)

$$LH_{a,\mu,\alpha} = \{\varphi : \varphi \in E(\Omega_1);$$

$$\gamma^{\mu}_{a',k',q',l}(\varphi) \equiv \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_q A^k k^{k\alpha}\}, \Delta = x^{-1} D_x.$$

The space $LH_{a,\mu,\alpha}$ is also defined as

$$LH_{a,\mu,\alpha} = \bigcup_{m=1}^{\infty} LH_{a,\mu,\alpha,m} \text{ (see Tiwari[8], p. 74)}$$

where $LH_{a,\mu,\alpha,m}$ is defined as (Tiwari[8], p. 51).

$$LH_{a,\mu,\alpha,m} = \{\varphi : \varphi \in LH_{a,\mu,\alpha};$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right.\right.$$

$$\left.\left. \leqslant C_{q\delta} (m + \delta)^k k^{k\alpha} \right\}$$

where $\Delta = x^{-1} D_x$ and δ is any number > 0.

The space $LH^{\beta}_{a,\mu}$: It is given by (See Tiwari[4], p. 1534 or Tiwari[8], p. 50).

$$LH^{\beta}_{a,\mu} = \{\varphi : \varphi \in E(\Omega_1);$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_k B^q q^{q\beta}\}.$$

The space $LH^{\beta}_{a,\mu}$ is also given by (For proof see Appendix)

$$LH^{\beta}_{a,\mu} = \bigcup_{n=1}^{\infty} LH^{\beta,n}_{a,\mu}$$

where $LH^{\beta,n}_{a,\mu}$ is defined by (Tiwari[8], p. 52)

$$LH^{\beta,n}_{a,\mu} = \left\{ \varphi : \varphi \in LH^{\beta}_{a,\mu} ; \right.$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\left. \leqslant C_{k\eta} (n + \eta)^q q^{q\beta} \right\}$$

where η is any number greater than 0.

The space $LH^{\beta}_{a,\mu,\alpha}$: This space is defined as (see Tiwari[4], p. 1535 or Tiwari[8], p. 50).

$$LH^{\beta}_{a,\mu,\alpha} \left\{ \varphi : \varphi \in E(\Omega_1); \right.$$

(equation continued on p. 927)

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x) \right\} \right.$$

$$\left. \leqslant CA^k B^q k^{k\alpha} q^{q\beta} \right\}.$$

Further the space $LH^{\beta}_{a,\mu,\alpha}$ is also defined as

$$LH^{\beta}_{a,\mu,\alpha} = \bigcup_{m,n=1}^{\infty} LH^{\beta,n}_{a,\mu,\alpha,m} \quad \text{(For proof see Appendix)}$$

where $LH^{\beta,n}_{a,\mu,\alpha,m}$ is defined as (see Tiwari[3], p. 52)

$$LH^{\beta,n}_{a,\mu,\alpha,m} = \left\{ \varphi : \varphi \in LH^{\beta}_{a,\mu,\alpha}\ ; \right.$$

$$\gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x) \right\} \right|$$

$$\left. \leqslant C_{\delta\eta} (m+\delta)^{\kappa} (n+\eta)^q k^{k\alpha} q^{q\beta} \right\}.$$

$\delta, \eta > 0$.

The author (Tiwari[4], p. 1535 or Tiwari[8], p. 61) proved that

$$LH^{\beta}_{a,\mu,\alpha} \subset \left(LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu} \right).$$

The aim of the present paper is to prove

$\left(LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu} \right) \subset LH^{\beta}_{a,\mu,\alpha}$ which together with the help of the result

$$LH^{\beta}_{a,\mu,\alpha} \subset \left(LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu} \right)$$

gives the equality of the spaces $LH^{\beta}_{a,\mu,\alpha}$ and $\left(LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu} \right)$.

2. Equality of Spaces $LH^{\beta}_{a,\mu,\alpha}$ and $\left(LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu} \right)$

Theorem 2.1— For $\alpha \geqslant 0$, $\beta > 0$.

$$LH^{\beta}_{a,\mu,\alpha} = \left(LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu} \right).$$

PROOF : We need prove only that

$$\left(LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu} \right) \subset LH^{\beta}_{a,\mu,\alpha}.$$

Let

$$\varphi \in LH_{a,\mu,\alpha} \cap LH^{\beta}_{a,\mu}.$$

But

$$LH_{a,\mu,\alpha} = \bigcup_{m=1}^{\infty} LH_{a,\mu,\alpha,m}.$$

Hence

$$\varphi \in LH_{a,\mu,\alpha,m} \text{ for some } m \geqslant 1.$$

Also since $LH^{\beta}_{a,\mu} = \bigcup_{n=1}^{\infty} LH^{\beta,n}_{a,\mu}$

$$\varphi \in LH^{\beta,n}_{a,\mu} \text{ for some } n \geqslant 1.$$

From the definition of the spaces $LH_{a,\mu,\alpha,m}$ and $LH^{\beta,n}_{a,\mu}$, we get

$$\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x) \right\} \right| \leqslant C_{q\delta} (m+\delta)^k k^{k\alpha} \quad \ldots(2.1)$$

and

$$\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x) \right\} \right| \leqslant C_{k\eta} (n+\eta)^q q^{q\beta}. \quad (2.2)$$

Since

$$m \geqslant 1, n \geqslant 1; \eta, \delta > 0, k^{k\alpha} \geqslant 1, q^{q\beta} \geqslant 1.$$

We get from (2.1) and (2.2)

$$\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right| \leqslant C_{q\delta} (m+\delta)^k (n+\eta^q) k^{k\alpha} q^{q\beta} \quad \ldots(2.3)$$

and

$$\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi \right\} \right| \leqslant C_{k\eta} (m+\delta)^k (n+\eta)^q \times k^{k\alpha} q^{q\beta}. \quad \ldots(2.4)$$

From (2.3 and (2.4) we have

$$\frac{\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x) \right\} \right|}{(m+\delta)^k (n+\eta)^q k^{k\alpha} q^{q\beta}} \leqslant C_{q\delta}$$

and

$$\frac{\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x) \right\} \right|}{(m+\delta)^k (n+\eta)^q k^{k\alpha} q^{q\beta}} \leqslant C_{k\eta}.$$

Above inequalities give us

$$\sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t,x) \right\} \right|$$

$$\leqslant C_{\delta\eta} (m+\delta)^k (n+\eta)^q k^{k\alpha} q^{q\beta}.$$

Hence

$$\varphi \in LH_{a,\mu,\alpha,m}^{\beta,n} \text{ for some } m, n \geqslant 1$$

But since

$$LH_{a,\mu,\alpha}^{\beta} = \cup LH_{a,\mu,\alpha,m}^{\beta,n}, \text{ we have}$$

$$\varphi \in LH_{a,\mu,\alpha}^{\beta}.$$

Consequently $\left(LH_{a,\mu,\alpha} \cap LH_{a,\mu}^{\beta} \right) \subset LH_{a,\mu,\alpha}^{\beta}$ and our proof is complete.

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APPENDIX

Proof of the result $LH^{\beta}_{a,\mu} = \bigcup_{n=1}^{\infty} LH^{\beta,n}_{a,\mu}$.

Let $\varphi \in \left(\bigcup_{n=1}^{\infty} LH^{\beta,n}_{a,\mu} \right) \Rightarrow \varphi \in LH^{\beta,n_0}_{a,\mu}$ for some n_0

$$\Rightarrow \gamma^{\mu}_{a,q,k,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_{k\eta} (n_0 + \eta)^q q^{q\beta}$$

$$\leqslant C_k \frac{C_{k\eta}}{C_k} (n_0 + \eta)^q q^{q\beta}$$

$$\leqslant C_k B^q q^{q\beta}$$

$$\Rightarrow \varphi \in LH^{\beta}_{a,\mu} \text{ and hence } \bigcup_{n=1}^{\infty} LH^{\beta,n}_{a,\mu} \subset LH^{\beta}_{a,\mu}.$$

Conversely Let

$$\varphi \in LH^{\beta}_{a,\mu}$$

$$\Rightarrow \gamma^{\mu}_{a,k,q,l}(\varphi) = \sup_{\Omega_1} \left| x^k e^{at} D_t^l \Delta^q \left\{ x^{-\left(\frac{2\mu+1}{2}\right)} \varphi(t, x) \right\} \right|$$

$$\leqslant C_k B^q q^{q\beta}.$$

Choose, an integer $n = n_B$ and $a\ \eta > 0$ such that

$$C_k B^q \leqslant C_{k\eta} (n + \eta)^q.$$

Then we immediately get $\varphi \in LH^{\beta,n}_{a,\mu}$ implying that $LH^{\beta}_{a,\mu} \subset \cup \left(LH^{\beta,n}_{a,\mu} \right)$

Similarly we can prove $LH^{\beta}_{a,\mu,\alpha} = \bigcup_{m,n=1}^{\infty} \left(LH^{\beta,n}_{a,\mu,\alpha,m} \right)$.