Indian J. pure appl. Math., 17 (7) : 943-956, July 1986

THIN CIRCULAR PLATE UNDER HYDROSTATIC PRESSURE OVER A CONCENTRIC ELLIPSE AND SUPPORTED AT FOUR POINTS

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(*Received* 8 *April* 1985; *after revision* 16 *December* 1985)

Complex variable methods are applied to derive expressions for the small deflections of a thin isotropic circular plate subject to normal hydrostatic loading over the area of a concentric ellipse and supported by four columns at the corners of a rectangle whose sides are parallel to the axes of the ellipse. Formulae are obtained for the moments and shears on the free edge and at the centre. Special and limiting cases are discussed.

1. INTRODUCTION

The transverse flexure of thin circular plates under various distributions of normal loadings and diverse boundary conditions has been studied by many authors. When the edge is elastically restrained and the circular plate is normally loaded over an eccentric circular patch the solutions were obtained by Bassali and Dawoud[1] and Bassali[2] (see also the References cited in these papers). Complex variable methods were applied by Bassali and Barsoum[3,4] to establish the appropriate solutions for a circular plate under the same loadings and supported along a concentric circle. Bassali and Nassif[5] and Bassali[6] used the method of complex potentials to deal with the elastically restrained circular plate under uniform and hydrostatic loadings over the area of a concentric ellipse. The bending of a circular plate having a free edge, supported at a discrete number of interior points and normally loaded over an eccentric circle was studied by the author[7,8]. Circular plates on multipoint supports have also been discussed by many[7-15]. In a recent paper the author[16] solved the problem of a circular plate which is uniformly loaded over a concentric ellipse and supported by four concentrated forces at the vertices of a rectangle whose sides are parallel to the axes of the ellipse. In this paper the same problem is treated but with linearly varying pressure over the area of of the ellipse.

2. FUNDAMENTAL EQUATIONS

Let C denote the boundary of a thin circular plate of centre 0, radius c, small constant thickness and flexural rigidity D. According to the Kirchhoff-Love small bending theory of normally loaded thin elastic slabs, the deflection w, measured positively

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downwards, at any point $z = x + iy = re^{i\theta}$ of the mid-plane of the plate satisfies the biharmonic equation

$$D\nabla^4 w = p(z, \bar{z}) \qquad \ldots(1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2} \qquad \ldots(2)$$

and $p(z, \bar{z})$ is the normal load intensity at the point z. The general solution of (1) is

$$w = 2\ \mathrm{Re}\,[\bar{z}\,\Omega(z) + \omega(z)] + W(z, \bar{z}) \qquad \ldots(3)$$

where $\Omega(z)$, $\omega(z)$ are functions of z which are analytic in the region occupied by the plate and $W(z, \bar{z})$ is a particular integral of (1). With the usual notation for the moments and shears we have[7]

$$M_r + M_\theta = -4(1 + \nu)\,D\,[2\ \mathrm{Re}\,\Omega' + \partial^2 W/\partial z \partial \bar{z}] \qquad \ldots(4a)$$

$$M_r - M_\theta + 2iM_{r\theta} = -4(1 - \nu)\,D\,[z\Omega'' + z^2(\omega'' + W'')/r^2] \qquad \ldots(4b)$$

$$Q_r - iQ_\theta = -8Dz\,[\Omega'' + \partial^3 W/\partial z^2\,\partial \bar{z}]/r \qquad \ldots(5)$$

where ν is Poisson's ratio for the material of the plate and accents denote differentiation with respect to z. In terms of polar coordinates and using the notations $d = \partial/\partial r$, $d' = \partial/\partial\theta$ we have[17]

$$M_r = -D\,(d^2 + \nu r^{-1} d + \nu r^{-2} d'^2)\,w = -D\,\{(1 - \nu)\,d^2 + \nu\nabla^2\}\,w \qquad \ldots(6a)$$

$$M_\theta = -D\,(\nu d^2 + r^{-1} d + r^{-2} d'^2)\,w = -D\,\{(\nu - 1)\,d^2 + \nabla^2\}\,w \qquad \ldots(6b)$$

$$M_{r\theta} = (1 - \nu)\,Dr^{-1}(d - r^{-1})\,d'\,w \qquad \ldots(7)$$

$$Q_r = -Dd\,(\nabla^2 w) \qquad \ldots(8a)$$

$$Q_\theta = -Dr^{-1} d'\,(\nabla^2 w). \qquad \ldots(8b)$$

3. Boundary Conditions and Boundary Values of Moments and Shears along a Free Circular Edge

The conditions for the circular edge C to be free are

$$(M_r)_{r=c} = 0 \qquad \ldots(9a)$$

$$\left(Q_r - \frac{1}{r}\,\frac{\partial M_{r\theta}}{\partial \theta}\right)_{r=c} = 0. \qquad \ldots(9b)$$

Substituting from (6a), (7), (8a) and (2) we get

$$[f_r(d, d')\,w]_{r=c} = 0 \qquad \ldots(10a)$$

$$[F_r(d, d')\, w]_{r=c} = 0 \qquad \ldots(10b)$$

where

$$f_r(d, d') = d^2 + \nu r^{-1} d + \nu r^{-2} d'^2 \qquad \ldots(11a)$$

$$F_r(d, d') = d^3 + r^{-1} d^2 - r^{-2} d + (2 - \nu) r^{-2} dd'^2 + (\nu - 3) r^{-3} d'^2. \qquad \ldots(11b)$$

From (6a) and (9a) we deduce that

$$\nu (\nabla^2 w)_{r=c} = (\nu - 1)(d^2 w)_{r=c}, \quad \nu (d' \nabla^2 w)_{r=c} = (\nu - 1)(d' d^2 w)_{r=c}.$$

Equations (6b) and (8b) now give

$$(M_\theta)_{r=c} = \frac{1 - \nu^2}{\nu} D (d^2 w)_{r=c} \qquad \ldots(12a)$$

$$(Q_\theta)_{r=c} = \frac{1 - \nu}{\nu c} D (d' d^2 w)_{r=c} \qquad \ldots(12b)$$

from which we obtain the useful relation

$$(d' M_\theta)_{r=c} = (1 + \nu) c (Q_\theta)_{r=c}. \qquad \ldots(13)$$

Equations (9b) and (13) enable us to compute the periphery values of the shears from the moment values.

4. Statement of the Problem

Let Γ denote the boundary of the ellipse

$$x^2/a^2 + y^2/b^2 = 1 \ (0 \leqslant b \leqslant a < c) \qquad \ldots(14)$$

and let the indices 1, 2 refer to the regions inside Γ and between Γ and C, respectively. The problem to be solved is to determine the deflection surface of the circular plate under the following conditions :

(1) The boundary C of the plate is free.

(2) The plate is subject to normal loading of intensities

$$p_1(x, y) = p_0 + p_0' x + p_0'' y \text{ over region 1, } p_2(x, y) = 0 \text{ over region 2,} \qquad \ldots(15)$$

where p_0, p_0' and p_0'' are constants. (See Fig. 1.)

(3) The plate is supported by four columns at the points P_λ ($z_\lambda = se^{i\gamma_\lambda}$, $\lambda = 1, 2, 3, 4$) where $0 < s < c$ and $\gamma_1 = \gamma$, $\gamma_2 = \pi - \gamma$, $\gamma_3 = -\gamma$, $\gamma_4 = \gamma - \pi$, $0 \leqslant \gamma < \pi/2$.

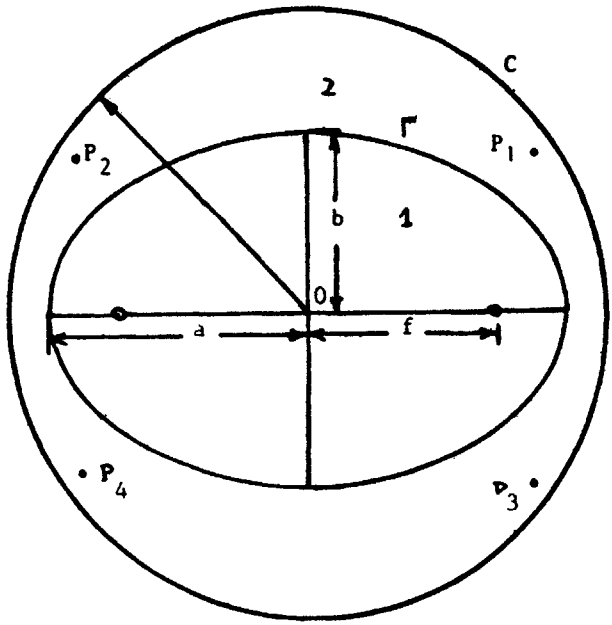

FIG. 1.

The deflection w vanishes at the four points of support. Applying the principle of superposition we see that $w = w_0 + w_0' + w_0''$, where w_0, w_0' and w_0'' are the deflections which correspond to three cases $p_1 = p_0$, $p_2 = 0$; $p_1 = p_0' x$, $p_2 = 0$ and $p_1 = p_0'' y$, $p_2 = 0$, respectively. In the first case symmetry with respect to both axes shows that reactions, measured positively upwards, at the four points of support are equal and each $= N = \pi p_0 ab/4$. In the second case there is symmetry with respect to the x-axis and antisymmetry with respect to the y-axis. If N', $-N'$, N' and $-N'$ are the reactions at P_1, P_2, P_3 and P_4, respectively, then the equilibrium of the plate gives $N' = M'/(4s\cos\gamma)$, $\gamma \neq \pi/2$, where $M' = \pi p_0' ba^3/4$ is the moment of the load about the y-axis. In the third case the problem is symmetric with respect to the y-axis and antisymmetric with respect to the x-axis. If N'', $N'' - N''$, $-N''$ are the reactions at the support points then $N'' = M''/s(4\sin\gamma)$, $\gamma \neq 0$, where $M'' = p_0'' ab^3/4$ is the moment of the load about the x-axis. Since the deflection w_0 was obtained before (Bassali[18]) and since the second and third cases are similar we now deal only with the second case.

5. METHOD AND SOLUTION

With a slight change in notation we assume that

$$p_1 = p_0 x = \tfrac{1}{2} p_0 (z + \bar{z}), \quad p_2 = 0 \qquad \ldots(16)$$

and the particular integrals may be taken as

$$W_1(z, \bar{z}) = \frac{p_0}{384D} z^2 \bar{z}^2 (z + \bar{z}),\ W_2(z, \bar{z}) = 0. \qquad ...(17)$$

The reactions at the four points of support P_1, P_2 P_3 and P_4 are N, $-N$, N and $-N$, respectively where

$$N = M/(4s \cos \gamma),\ \gamma \neq \pi/2,\ M = \pi p_0 b a^3/4. \qquad ...(18)$$

For $\gamma = 0$ we have only two supports at $(\pm s, 0)$ at the reactions equal $\pm M/2s$.

Symmetry considerations show that it is sufficient to obtain the deflection w at any point z in the positive quadrant of the plate. It is known that the continuity requirements for the deflections, slopes, moments and shears at any point on Γ lead to

$$[w]_1^2 = [\partial w/\partial z]_1^2 = [\partial^2 w/\partial z \partial \bar{z}]_1^2 = [\partial^3 w/\partial z^2 \partial \bar{z}]_1^2 = 0$$

along Γ and it was proved[6] that these transition conditions along Γ are satisfied by*

$$k\,[\Omega(z)]_1^2 = \frac{1}{ba^3}\left[6(1-\alpha)a^4 - 6\alpha b^2 z^2 + \left(2\alpha^2 - \alpha - \frac{1}{4}\right) z^4\right] - 3 \ln \frac{z+Z}{a+b} + \frac{zZ}{f^2}\left(5 - \frac{2z^2}{f^2}\right) \qquad ...(19a)$$

$$k[\omega(z)]_1^2 = \frac{z^3}{ba^3}\left[2\alpha(4\alpha - 1) b^2 - \frac{1}{5}\left(\frac{1}{4} + 3\alpha - 18\alpha^2 + 16\alpha^3\right) z^2\right] - 3z \ln \frac{z+Z}{a+b} + [3.2\alpha + (5 - 6.4\alpha) z^2/f^2 + (3.2\alpha - 2) z^4/f^4]\, Z \qquad ...(19b)$$

where

$$Z = \sqrt{(z^2 - f^2)},\ k = 48\pi D/M,\ \alpha = a^2/f^2 \qquad ...(20)$$

Introducing (17) and (19a, b) in (3) we get

$$k\,[w]_1^2 = 2\,\mathrm{Re}\left\{\left[3.2\alpha + \frac{5r^2}{f^2} + \left(5 - 6.4\alpha - \frac{2r^2}{f^2}\right)\frac{z^2}{f^2} + (3.2\alpha - 2) \times \frac{z^4}{f^4}\right] Z - 6r \cos\theta \ln \frac{z+Z}{a+b} + \frac{z}{ba^3}\left[6(1-\alpha) a^4 - 6\alpha b^2 r^2 - \frac{1}{2} r^4 + \left(2\alpha(4\alpha - 1) b^2 + \left(2\alpha^2 - \alpha - \frac{1}{4}\right) r^2\right) z^2 - \frac{1}{5}\left(\frac{1}{4} + 3\alpha - 18\alpha^2 + 16\alpha^3\right) z^4\right]\right\}. \qquad ...(21)$$

*Notice that in this paper the deflection is measured positively downwards and the last term in eqn. (24b) is combined with the last term in eqn. (24a). (See Bassli[6], p. 114).

If ϕ is the eccentric angle of any point z on Γ then

$$z = a\cos\phi + ib\sin\phi,\ Z = b\cos\phi + ia\sin\phi$$

and it is verified that $[w]_1^2$ vanishes on Γ as expected. Since Z is uniform in region 2 and not uniform in region 1 it follows that the terms involving Z in (21) should appear in w_2. It is known that the singular part of the deflection w at any point P near a downward concentrated force F acting at Q is

$$w = \frac{F}{8\pi D} R^2 \ln R \qquad \ldots(22)$$

where $R = PQ$. Guided by these remarks and using (21, (18) we assume that

$$kw_2 = -12s\cos\theta \ln\left|\frac{z+Z}{a+b}\right| + 2\,\mathrm{Re}\left\{3.2\alpha + \frac{5r^2}{f^2} + \left(5 - 6.4\alpha - \frac{2r^2}{f^2}\right)\frac{z^2}{f^2} + (3.2\alpha - 2)\frac{z^4}{f^4}\right\} Z + \frac{3S}{2s}\sec\gamma + \sum_0^\infty (A_n + C_n r^2) r^{2n+1}\cos(2n+1)\theta \qquad \ldots(23a)$$

$$kw_1 = \frac{1}{ba^3}[r\{12(\alpha-1)a^4 + 12\alpha b^2 r^2 + r^4\}\cos\theta + r^3\{4\alpha(1-4\alpha)b^2 + \left(\frac{1}{2} + 2\alpha - 4\alpha^2\right) r^2\}\cos 3\theta + \frac{2}{5} r^5\left(\frac{1}{4} + 3\alpha - 18\alpha^2 + 16\alpha^3\right)\cos 5\theta] + \frac{3S}{2s}\sec\gamma + \sum_0^\infty (A_n + C_n r^2) r^{2n+1}\cos(2n+1)\theta \qquad \ldots(23b)$$

where

$$S = \sum_{\lambda=1}^{4} (-1)^\lambda R_\lambda^2 \ln R_\lambda \qquad \ldots(24)$$

$$\left.\begin{aligned} R_1^2 &= r^2 + s^2 - 2sr\cos(\theta-\gamma),\ R_2^2 = r^2 + s^2 + 2sr\cos(\theta+\gamma) \\ R_3^2 &= r^2 + s^2 - 2sr\cos(\theta+\gamma),\ R_4^2 = r^2 + s^2 + 2sr\cos(\theta-\gamma) \end{aligned}\right\} \qquad \ldots(25)$$

and A_n, C_n ($n = 0, 1, 2, \ldots$) are real constants to be determined for the boundary conditions (10a, b) and the condition that the deflection is zero at any of the support points. The terms containing Z in (23a) can be expressed in finite forms by using the formulae

$$| z + Z | = [r^2 + T^2 + \sqrt{2}r \sqrt{(T^2 + r^2 - f^2 \cos 2\theta)}]^{1/2} \quad ...(26)$$

$$\text{Re}\,(z^n Z) = \frac{r^n}{\sqrt{2}}[\cos n\theta \sqrt{(T^2 + r^2 \cos 2\theta - f^2)} - \sin n\theta \sqrt{(T^2 + f^2 - r^2 \cos 2\theta)}] \; (n = 0, 1, 2, \ldots) \quad ...(27)$$

where

$$T^4 = r^4 + f^4 - 2f^2 r^2 \cos 2\theta. \quad ...(28)$$

These formulae are valid whether $r \geqslant f$ or $r \leqslant f$. For $r = f$ they simplify to

$$| z + Z | = f(\sqrt{1 + \sin\theta} + \sqrt{\sin\theta}) \quad ...(29)$$

$$\text{Re}\,(z^n Z) = = fr^n \sqrt{\sin\theta}\,(\cos n\theta \sqrt{1 - \sin n\theta} - \sin n\theta \sqrt{1 + \sin\theta}). \quad ...(30)$$

To determine the constants A_n $(n = 1, 2, \ldots)$ and C_n $(n = 0, 1, 2, \ldots)$ we need to expand the functions containing Z in (23a) in terms of biharmonic functions of the types $r^{\pm n} \begin{matrix}\cos \\ \sin\end{matrix} n\theta$ and $r^{2\pm n} \begin{matrix}\cos \\ \sin\end{matrix} n\theta$. Noting that $\frac{d}{dz}(z + Z) = (z^2 - f^2)^{-1/2}$, expanding by the binomial theorem and integrating term by term we arrive at

$$\ln | z + Z | = \ln 2r - \sum_1^\infty \frac{(f/2r)^{2n}}{2n} \delta_n \cos 2n\theta \; (r \geqslant f) \quad ...(31a)$$

$$\ln | z + Z | = \ln f + 2 \sum_0^\infty \frac{(r/2f)^{2n+1}}{2n+1} \delta_n \sin (2n + 1)\theta \; (r \leqslant f) \quad ...(31b)$$

where $\delta_n = \binom{2n}{n}$. For $r = f$ it can be shown that the right sides of (31a, b) converge to $\ln | z + Z | = \ln f + \ln(\sqrt{1 + \sin\theta} + \sqrt{\sin\theta})$ Substituting fron (31a, b) in (23a) and obtaining the real parts of the remaining terms containing Z after expanding by the binomial theorem we get after some algebraic manipulation

$$kw_2 = -12r \cos\theta \ln \frac{2r}{a + b} + 3r\left\{4\alpha - 3 + \frac{4r^2}{f^2} + \frac{1}{6}(1 - 4\alpha)\frac{f^2}{r^2}\right\} \times \cos\theta + \frac{4r^3}{f^2}\left\{\left(3 - 4\alpha - \frac{r^2}{f^2}\right)\cos 3\theta + (1.6\alpha - 1)\frac{r^2}{f^2}\cos 5\theta\right\}$$

$$+ \frac{3S}{2s}\sec\gamma + 6f \sum_1^\infty \frac{(f/2r)^{2n+1}}{(n+1)\,n+2)}\delta_n \left(\lambda_n + \frac{2r^2}{nf^2}\right)\cos(2n$$

(equation continued on p. 950)

$$+ 1)\,\theta + \sum_{0}^{\infty} (A_n + C_n r^2)\, r^{2n+1} \cos (2n + 1)\,\theta \; (r \geqslant f) \quad \ldots(32a)$$

$$kw_2 = -12r \cos\theta \ln \frac{f}{a+b} + 8\,(r^2 + 2a^2 - 2f^2)\,\frac{r^2}{f^3} \sin 2\theta$$

$$+ \frac{3S}{2s} \sec\gamma + 48 f \sum_{2}^{\infty} \frac{(r/2f)^{2n}}{(2n-1)\,(2n-3)}\, \delta_n \left(2\mu_n - \frac{r^2/f^2}{2n+1}\right) \sin 2n\theta$$

$$+ \sum_{0}^{\infty} (A_n + C_n r^2)\, r^{2n+1} \cos (2n + 1)\,\theta \; (r \leqslant f) \quad \ldots(32b)$$

where

$$\lambda_n = \frac{1}{n+3}\left(\frac{2n+1}{n+1} - 4\alpha\right), \; \mu_n = \frac{1}{2n-5}\left(\alpha - \frac{n}{2n-1}\right). \quad \ldots(33)$$

It is easily seen that points in region 2 at which $r \leqslant f$ exist only if $r \geqslant b$ i. e. if the eccentricity of the ellipse $\geqslant \sqrt{2}/2$. The sum S of the logarithmic terms defined by (24) and (25) can be expanded in terms of biharmonic functions of (r, θ). For $r \geqslant s$ we have

$$R_\lambda^2 \ln R_\lambda = (r^2 + s^2) \ln r + s^2 - sr\left(1 + 2 \ln r + \frac{s^2}{2r^2}\right) \cos(\theta - \gamma_\lambda)$$

$$+ \sum_{2}^{\infty} \frac{1}{n}\left(\frac{r^2}{n-1} - \frac{s^2}{n+1}\right)\left(\frac{s}{r}\right)^n \cos n\,(\theta - \gamma_\lambda) \; (r \geqslant s)$$

$$\ldots(34a)$$

$$S = 4sr\left(1 + 2 \ln r + \frac{s^2}{2r^2}\right) \cos\gamma \cos\theta$$

$$+ 2 \sum_{1}^{\infty} \left(\frac{s^2}{n+1} - \frac{r^2}{n}\right) \frac{(s/r)^{2n+1}}{2n+1} \cos (2n + 1)\,\gamma \cos (2n + 1)\,\theta$$

$$(r \geqslant s). \quad \ldots(34b)$$

If $r \leqslant s$ we interchange r, s in (34a,b). Substituting for S from (34b) in (32a) we find that kw_2 can be written in the form

$$kw_2 = \sum_{0}^{\infty} L_n(r) \cos (2n + 1)\,\theta \quad \ldots(35)$$

where $r \geqslant$ the greater of f and s

$$L_0(r) = A_0' r + B_0' r^{-1} + C_0' r^3 \quad \ldots(36a)$$

$$L_n(r) = A'_n r^{2n+1} + B'_n r^{2n-1} + C'_n r^{2n+3} + D'_n r^{1-2n} \ (n \geqslant 1) \quad \ldots(36b)$$

$$A'_0 = A_0 + 3\left(4\alpha - 1 + 4 \ln \frac{a+b}{2}\right), B'_0 = \tfrac{1}{2}(6s^2 - 3a^2 - b^2),$$

$$C'_0 = C_0 + 12/f^2 \quad \ldots(37)$$

$$A'_1 = A_1 + 4(3 - 4\alpha)/f^2, \ C'_1 = C_1 - 4/f^4, \ A'_2 = A_2 + (6.4\alpha - 4)/f^4 \quad \ldots(38)$$

$$A'_n = A_n \ (n \geqslant 3), C'_n = C_n \ (n > 2) \quad \ldots(39)$$

$$B'_n = \frac{3}{n+1}\left[\frac{s^{2n+2}\cos(2n+1)\gamma}{(2n+1)\cos\gamma} + \frac{4(f/2)^{2n+2}}{n+2}\delta_n \lambda_n\right](n \geqslant 1) \quad \ldots(40)$$

$$D'_n = \frac{3}{n}\left[-\frac{s^{2n}\cos(2n+1)\gamma}{(2n+1)\cos\gamma} + \frac{2(f/2)^{2n}}{(n+1)(n+2)}\delta_n\right](n \geqslant 1). \quad \ldots(41)$$

Applying the differential operators (11a, b) to the functions in eqn. (35) we find

$$f_r(d,d')\{L_0(r)\cos\theta\} = 2\sigma r\left(B'_0 r^{-4} + \kappa C'_0\right)\cos\theta \quad \ldots(42a)$$

$$F_r(d,d')\{L_0(r)\cos\theta\} = 2\sigma\left(B'_0 r^{-4} + \kappa C'_0\right)\cos\theta \quad \ldots(42b)$$

$$f_r(d,d')\{L_n(r)\cos(2n+1)\theta\} = 2(2n+1)\sigma r^{-1}\left[n A'_n r^{2n} + (n+1) B'_n r^{-2n-2} + \frac{(n+1)(2n-\kappa)}{2n+1} C'_n r^{2n+2} + \frac{n(2n+2-\kappa)}{2n+1} D'_n r^{-2n}\right](n > 1) \quad \ldots(43a)$$

$$F_r(d,d')\{L_n(r)\cos(2n+1)\theta\} = -2(2n+1)^2\sigma r^{-2}\left[n A'_n r^{2n} - (n+1) B'_n r^{-2n-2} + \frac{(n+1)(2n-\kappa)}{2n+1} C'_n r^{2n+2} - \frac{n(2n+2+\kappa)}{2n+1} D'_n r^{-2n}\right](n \geqslant 1) \quad \ldots(43b)$$

where

$$\sigma = 1 - \nu, \kappa = (3 + \nu)/(1 - \nu). \qquad \ldots(44)$$

Introducing (35) in the boundary conditions (10a, b) using (42a, b), (43a,b), equating the coefficients of $\cos(2n + 1)\theta$ $(n = 0, 1, 2, \ldots)$ in the resulting identities to zero and solving the linear equations obtained we find

$$C_0' = \frac{1}{2\kappa C^4}(3a^2 + b^2 - 6s^2) \qquad \ldots(45a)$$

$$A_n' = \frac{6}{\kappa} c^{-2n}\left[\frac{u^{2n}\cos(2n+1)\gamma}{(2n+1)\cos\gamma}\left\{u^2 - \frac{\kappa^2 + 4n(n+1)}{2n(2n+1)}\right\}\right.$$
$$\left. + \frac{v^{2n}}{(n+1)(n+2)}\delta_n\left\{4(n+1)\nu^2\lambda_n + \frac{\kappa^2 + 4n(n+1)}{n(2n+3)}\right\}\right]$$
$$(n \geqslant 1) \qquad \ldots(45b)$$

$$C_n' = \frac{6}{\kappa} c^{-2n-2}\left[\frac{u^{2n}\cos(2n+1)\gamma}{(2n+1)\cos\gamma}\left\{1 - \frac{2n+1}{2n+2}u^2\right\}\right.$$
$$\left. - \frac{2v^{2n}}{(n+1)(n+2)}\delta_n\{1 + (2n+1)\nu^2\lambda_n\}\right] (n \geqslant 1) \qquad \ldots(45c)$$

where $\nu = f/2c$. When the values of C_0, A_1, C_1 and A_2 are replaced by their values in terms of C_0', A_1', C_1' and A_2' by means of (37) and (38) it is found that deflections (23 b) and (32a) become

$$kw_1 = 3r\left[1 + 4\ln\frac{2}{a+b} - \frac{4(r^2+a^2)}{a(a+b)} + \frac{r^4}{3ba^3}\right]\cos\theta$$
$$+ \frac{4(a-b)r^3}{a(a+b)^2}\left[1 - \frac{a+3b}{8ba^2}r^2\right]\cos 3\theta + \frac{(a-b)^2(5a+b)}{10ba^3(a+b)^3}$$
$$r^5\cos 5\theta + \frac{3S}{2s}\sec\gamma + \sum_0^\infty\left(A_n' + C_n' r^2\right)r^{2n+1}\cos(2n+1)\theta$$
$$\ldots(46a)$$

$$kw_2 = -6r\left[1 + 2\ln r + \frac{3a^2+b^2}{12r^2}\right]\cos\theta + \frac{3S}{2s}\sec\gamma$$
$$+ 6f\sum_1^\infty\frac{(f/2r)^{2n+1}}{(n+1)(n+2)}\delta_n\left(\lambda_n + \frac{2r^2}{nf^2}\right)\cos(2n+1).$$
$$+ \sum_0^\infty\left(A_n' + C_n' r^2\right)r^{2n+1}\cos(2n+1)\theta\,(r \geqslant f). \qquad \ldots(46b)$$

For points of region 2 at which $r \leqslant f$ ($f \geqslant b$), the deflection w_2 is given by (32b) where all the required constants are now determined except A_0'. Whether $r \geqslant f$ or $r \leqslant f$ eqns. (23a) and (26) – (30) can be used to find w_2. The value of A_0' depends on whether the supports lie in region 1 or region 2. In the first case $\cos^2\gamma/a^2 + \sin^2\gamma/b^2 < s^2$ ($s \neq 0$) and the vanishing of w_1 given by (46a) at a support point yields

$$A_0' = 6\left[\frac{2a^2 + 2s^2}{a(a+b)} + 2\ln\frac{a+b}{2} - \tfrac{1}{2} + \tan^2\gamma \ln\sin\gamma - \ln(4s^2\cos\gamma)\right] + s^2\left[\frac{6s^2 - 3a^2 - b^2}{2\kappa c^4} - \frac{s^2}{ba^3} - \frac{4(a-b)\sec\gamma}{a(a+b)^2}\left\{\left(1 - \frac{a+3b}{8ba^2}s^2\right)\cos 3\gamma + \frac{(a-b)(5+b)s^2}{40b(a+b)a^2}\cos 5\gamma\right\}\right]$$

$$- \sum_1^\infty \left(A_n' + C_n' s^2\right) s^{2n} \frac{\cos(2n+1)\gamma}{\cos\gamma}. \qquad \ldots(47)$$

If the supports lie in region 2 either (46b) or (32b) can be used to determine A_0' according as $s \geqslant f$ or $s \leqslant f$.

The moments and shears at any point of the plate can be found by introducing (46a,b), (32b) in (6a,b), (7), (8a,b) and noting that S has the expansion (34b) if $r \geqslant s$ and interchanging r, s in (34b) if $r \leqslant s$. It is found that $(M_r)_{r=c} = 0$ as it should, and

$$(M_\theta)_{r=c} = \frac{(1+\nu)M}{2\pi\kappa c}\left[\beta\cos\theta + \sum_1^\infty \frac{u^{2n}}{\cos\gamma}\left(u_2 - 1 - \frac{\kappa+1}{2n+1}\right)\cos(2n+1)\gamma\cos(2n+1)\theta + 2\sum_1^\infty \frac{v^{2n}}{n+2}\delta_n\left\{2 + \frac{\kappa}{n+1} + 2(2n+1)v^2\lambda_n\right\}\cos(2n+1)\theta\right]. \qquad \ldots(48a)$$

$$(M_{r\theta})_{r=c} = \frac{M}{2\pi\kappa c}\left[\beta\sin\theta + \sum_1^\infty \frac{u^{2n}}{\cos\gamma}\left(u^2 - 1 + \frac{\kappa-1}{2n+1}\right)\cos(2n+1)\gamma\sin(2n+1)\theta\right]$$

(*equation continued on p. 954*)

$$+ 2 \sum_{1}^{\infty} \frac{\nu^{2n}}{n+2} \delta_n \left\{ 2 - \frac{\kappa}{n+1} + 2(2n+1)\nu^2\lambda_n \right\}$$

$$\sin(2n+1)\theta] \qquad \ldots(48b)$$

$$(Q_r)_{r=c} = \frac{M}{2\pi\kappa c^2}[\beta \cos\theta + \sum_{1}^{\infty} \frac{u^{2n}}{\cos\gamma}\{\kappa - 1 + (2n+1)(u^2-1)\}$$

$$\cos(2n+1)\gamma \cos(2n+1)\theta$$

$$+ 2 \sum_{1}^{\infty} \frac{2n+1}{n+2} \nu^{2n} \delta_n \left\{ 2 - \frac{\kappa}{n+1} + (2n+1)\nu^2\lambda_n \right\}$$

$$\cos(2n+1)\theta] \qquad \ldots(49a)$$

$$(Q_\theta)_{r=c} = \frac{M}{2\pi\kappa c^2}\left[-\beta \sin\theta + \sum_{1}^{\infty} \frac{u^{2n}}{\cos\gamma}\{\kappa + 1 + (2n+1)(1-u^2)\}\right.$$

$$\cos(2n+1)\gamma \sin(2n+1)\theta$$

$$\left. -2 \sum_{1}^{\infty} \frac{2n+1}{n+2} \nu^{2n} \delta_n \left\{ 2 + \frac{\kappa}{n+1} + 2(2n+1)\nu^2\lambda_n \right\} \right.$$

$$\sin(2n+1)\theta] \qquad \ldots(49b)$$

where $\beta = u^2 - (3a^2 + b)/6c^2$. It is easily verified that (48a), (49b) satisfy (13) and (48b), (49a) satisfy (9b).

The convergence of all the infinite series involving δ_n in (31a,b), (32a,b), (46a,b), (48a,b), and (49a,b) follows from the fact that for large values of n we have $\delta_n/2^{2n} \simeq 1/\sqrt{\pi n}$. This can be easily deduced by applying Stirling's formula $n! \simeq \sqrt{2\pi n}\ (n/e)^n$ for large n. The infinite series containing γ in (48a,b) and (49a,b) can be summed up.

At the centre of the plate the moments vanis and the shears are given by

$$\frac{(Q_r)_0}{\cos\theta} = \frac{(Q_\theta)_0}{\sin\theta} = \frac{M}{2\pi}\left[\frac{4}{a(a+b)} - \frac{1}{s^2} + \frac{\beta}{\kappa c^2} \right]. \qquad \ldots(50)$$

7. Limiting Cases

The bending of an infinite plate acted upon by the load (16) over the area of the ellipse (14) and supported at the four points z_1, z_2, z_3 and z_4 can be investigated by letting $c \to \infty$ in the foregoing resuits. Eqns. (45) and (37) — (39) show that in this case we have

$$\left.\begin{aligned} &C_0' = 0,\ A_n' = C_n' = 0\ (n \geqslant 1) \\ &C_0 = -12/f^2,\ A_1 = 4\,(4\alpha - 3)/f^2,\ C_1 = 4/f^4, \\ &\qquad\qquad A_2 = (4 - 6.4\alpha)/f^4 \\ &A_n = 0\ (n \geqslant 3),\ C_n = 0\ (n \geqslant 2). \end{aligned}\right\} \quad \ldots(51)$$

The deflections w_1 and w_2 $(r \geqslant f)$ are then given by (46a, b) with $A_0'\ r\ \cos\theta$ as the only surviving term in the infinite series. At points where $r \leqslant f\,(f \geqslant b)$ the deflection w_2 is given by (32b) in which the infinite series reduces to

$$(A_0 + C_0 r^2)\ r\cos\theta + (A_1 + C_1 r^2)\ r^3 \cos 3\theta + A_2 r^5 \cos 5\theta \quad \ldots(52)$$

where C_0, A_1, C_1, A_2 are defined by (51) and A_0, A_0' are related by the first eqn. of (37). If the supports lie in the loaded region then A_0' is given by (47) in which all the terms of the infinite series vanish.

When the minor axis of the ellipse $\to 0$ the loaded patch reduces to a line loading extending along the x-axis from $x = -a$ to $x = a$. If $b \to 0$ and $p_0 \to \infty$ such that $2bp_0 \to p_1$ then the intensity of this line loading is $p_1\ x\ \sqrt{(1 - x^2/a^2)}$. Expressions for the deflection in this case can be obtained from those for region 2 in section 5 by setting $b = 0$, $f = a$ and $M = \frac{1}{8}\pi p_1\ a$.

Setting $b = a$, $f = 0$ in eqns. (45) – (50) we obtain the appropriate formulae corresponding to linearly varying load over a concentric circle.

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