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SOME FIXED POINT THEOREMS IN BANACH SPACE FOR COMMUTING MAPPINGS

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In this paper some common fixed point theorems have been established for a pair of commuting mappings, satisfying a new contractive type condition, on certain subsets of a Banach space. Our results improve the results of Khan[3] and Goebel-Zlotkiewick[2]. An illustrative example is given in support of our theorems.

In a recent paper, Khan[3] obtained a number of fixed point theorems for certain commuting mappings in Banach spaces. The results obtained by Khan[3] can be viewed as applications of those due to Goebel Zlotkiewicz[2] and Dotson[1] others are indeed generalization of results due to Shimi[4]. Khan[3] proved the following :

Theorem A—Let K be a closed and convex subset of Banach space X. Let $F : K \to K$, $G : K \to K$ satisfy the following conditions :

(a_1) F and G commute,

(a_2) $F^2 = I$ and $G^2 = I$, where I denotes the identity mapping

(a_3) $\|Fx - Fy\| \leqslant \alpha \| Gx - Gy\|$

for every $x, y \in K$ and $0 \leqslant \alpha < 2$. Then there exists at least one point $x_0 \in K$ such that $F(x_0) = G(x_0)$. Further, if $0 \leqslant \alpha < 1$, then x_0 is unique and $x_0 = F(x_0) = G(x_0)$.

The purpose of this paper is to improve upon the above result and establish some common fixed point theorems for a pair of mappings defined on certain subsets of a Banach space satisfying a new contractive type condition.

Our main results are established in the following theorems :

Theorem 1—Let M be a nonempty closed convex subset of a Banach space B. Let $F : M \to M$, $G : M \to M$ satisfy the following conditions :

(i) F and G commutes;

(ii) $F^2 = I$ and $G^2 = I$, where I denotes the identity mappings;

(iii) $\|Fx - Fy\|^2 \leqslant q \max \{\|Gx - Fx\| \|Gy - Fy\|, \|Gx - Fy\| \|Gy - Fx\|,$

$\|Gx - Fx\| \| Gy - Fx\|, \|Gx - Fy\| \|Gy - Fy\|\}$

for all $x, y \in M$, where $q \in (0, 1)$. Let $x_1 \in M$ be arbitrary, $t \in (0, 1)$ and $Gx_{n+1} = (1 - t) Gx_n + t Fx_n$ for each integer $n \geqslant 1$. If the sequence $\{Gx_n\}$ converges to a point $u \in M$, then u is the unique common fixed point of F and G.

PROOF : For each $n \geqslant 1$, we have $Gx_{n+1} = (1 - t) Gx_n + t Fx_n$

which implies

$$\|Gx_{n+1} - F G u\|^2 \leqslant [(1 - t) \| Gx_n - F G u\| + t\| Fx_n - F Gu\|]^2$$

$$\leqslant (1 - t)^2 \|Gx_n - F G u\|^2 + 2t (1 - t) \|Gx_n - F G u\| \|Fx_n$$

$$- F G u\| + t^2 \| Fx_n - F G u\|. \quad \ldots(1)$$

Since F and G satisty (iii), we have by using (ii)

$$\|Fx_n - F G u\|^2 \leqslant q \max \{\|G x_n - Fx_n\| \|G^2u - F Gu\| , \|Gx_n - FGu\|$$

$$\|G^2u - Fx_n\|, \|Gx_n - Fx_n\| \|G^2u - Fx_n\|,$$

$$\|Gx_n - FG_n\| \|G^2u - FGu\|\}$$

$$\leqslant q \max \{\|Gx_n - Fx_n\| \|u - FGu\|,$$

$$\|Gx_n - FGu\| [\|u - Gx_n\| + \|Gx_n - Fx_n\|],$$

$$\|Gx_n - Fx_n\| [\|u - Gx_n\| + \|Gx_n - Fx_n\|],$$

$$\|Gx_n - FGu \| \|u - FGu\|\}. \quad \ldots(2)$$

From (1) and (2) we obtain

$$\|Gx_{n+1} - FGu\|^2 \leqslant (1 - t)^2 \|Gx_n - FGu\|^2 + 2t (1 - t)\|Gx_n - FGu\|$$

$$[\|Fx_n - Gx_n\| + \|Gx_n - FGu\|] + t^2 q \max \{\|Gx_n - Fx_n\|$$

$$\|u - FGu\|, \|Gx_n - FGu\| [\|u - Gx_n\| + \|Gx_n - Fx_n\|],$$

$$\|Gx_n - Fx_n\| [\|u - Gx_n\| + \|Gx_n - Fx_n\|],$$

$$\|Gx_n - FGu\| \|u - FGu\|\}. \quad \ldots(3)$$

Since $\{Gx_n\}$ converges to u and $Gx_{n+1} - Gx_n = t (Fx_n - Gx_n)$, then $\{Fx_n - Gx_n\}$ tends to zero. By letting $n \to \infty$ in (3), we obtain

$$\|u - FGu\|^2 \leqslant (1 - t)^2 \|u - FGu\|^2 + 2t (1 - t) \|u - FGu\|^2$$

$$+ t^2 q \max \{0, 0, 0, \|u - FGu\|^2\}$$

$$= (1 - t)^2 \|u - FGu\|^2 + 2t (1 - t) \|u - FGu\|^2$$

$$+ t^2 q \| u - FGu\|^2$$

$$= \beta \|u - FGu\|^2$$

where $\beta = [1 - (1 - q)\, t^2]$. Since $\beta \in (0, 1)$, FG has at least one fixed point $u \in M$ i. e.

$$FGu = u \qquad \ldots(4)$$

Now using (ii), we have

$$Fu = F^2Gu = Gu. \qquad \ldots(5)$$

Again using (i), (ii), and (4) and (5), we have

$$\begin{aligned} \|u - Fu\|^2 &= \|F(Fu) - Fu\|^2 \\ &\leqslant q \max \{\|GFu - F^2u\| \|Gu - Fu\|, \|GFu - Fu\| \|Gu - F^2u\|, \\ &\qquad \|GFu - F^2u\| \|Gu - F^2u\|, \|GFu - Fu\| \|Gu - Fu\|\} \\ &= q \max \{0, \|u - Fu\|^2, 0, 0\} \\ &= q \|u - Fu\|^2. \end{aligned}$$

Since $q < 1$, it follows $Fu = u$ i. e. u is the fixed point of F, but $Fu = Gu$ and so we have $Gu = u$ i. e. u is the common fixed point of F and G. Now to show the uniqueness of u, let us consider v $(v \neq u)$ be another common fixed point of F and G.

$$\begin{aligned} \|u - v\|^2 &= \|F^2u - F^2v\|^2 = \|F(Fu) - F(Fv)\|^2 \\ &\leqslant q \max \{\|GFu - F^2u\| \|GFv - F^2v\|, \|GFu - F^2v\| \|GFv - F^2u\|, \\ &\qquad \|GFu - F^2u\| \|GFv - F^2u\|, \|GFu - F^2v\| \|GFv - F^2v\|\} \\ &= q \max \{0, \|u - v\|^2, 0, 0\} \\ &= q \|u - v\|^2 \end{aligned}$$

Since $q \in (0, 1)$, it follows $u = v$ proving the uniqueness of u. This completes the proof of the theorem.

Now we wish to investigate the solvability of certain non-linear functional equations in a Banach space.

Theorem 2—Let $\{f_n\}$ be a sequence of elements in a Banach space B. Let v_n be the unique solution of the equation $u - FGu = f_n$, where F and G are mappings of B into itself satisty conditions (i), (ii) and (iii) of Theorem 1.

If $\|f_n\| \to 0$ as $n \to \infty$, then the sequence $\{v_n\}$ converges to the solution of the equation $u = Fu = Gu$.

PROOF : By (iii), we have for $m > n$

$$\begin{aligned} \|v_n - v_m\|^2 &\leqslant [\|v_n - FGv_n\| + \|FGv_n - FGv_m\| + \|FGv_m - v_m\|]^2 \\ &\leqslant \{\|v_n - FGv_n\| + \|FGv_m - v_m\|\}^2 + 2\{\|v_n - FGv_n\| + \|FGv_m \end{aligned}$$

(equation continued on p. 972)

$$-v_m\|\}\,[\|FGv_n - v_n\| + \|v_n - v_m\| + \|v_m - FGv_m\|] + q \max$$
$$\{\|u_n - FGv_n\|\,\|v_m - FGv_m\|, [\|v_n - v_m\| + \|v_m - FGv_m\|]$$
$$\times [\|v_m - v_n\| + \|v_n - FGv_n\|],$$
$$\|v_n - FGv_n\|\,[\|v_m - v_n\| + \|v_n - FGv_n\|]$$
$$[\|v_n - v_m\| + \|v_m - FGv_m\|]\,\|v_m - FGv_m\|\}.$$

On letting $n \to \infty$ and using the hypothesis, we obtain

$$\|v_n - v_m\|^2 \leqslant q\,\|v_n - v_m\|^2$$

which is impossible since $q \in (0,1)$. It follows therefore that $\{v_n\}$ is a Cauchy sequence, Hence it converges to v, say. Also it is an easy excercise to show that

$$\|v - FGv\|^2 \leqslant q\,\|v - FGv\|^2$$

which implies $v = FGv$. Now $Fv = F^2Gv = Gv$. The rest of the proof of this theorem goes in a similar fashion as that of Theorem 1, so we omit the proof.

Finally, we furnish an example to discuss the validity of the hypothesis and degree of generality of Theorem 1.

Example 1—Let $B = R^3$, where R^3 is the set of all 3-tuples $x = (x(1), x(2), x(3))$ of real numbers and the norm $\|x\|$ is defined by

$$\|x\| = \Big(\sum_{i=1}^{3} |\,x(i)\,|^2\Big)^{1/2}, \qquad x \in R^3.$$

Further, let $M = \{x : \|x - 0\| \leqslant 1,\ 0,\ x \in R^3\}$ and we define mapping F and G of M into itself such that for any arbitrary $x = (x(1), x(2), x(3)) \in M$

$$Fx = (-x(1), -x(2), -x(3)) \text{ and } Gx = (x(2), x(1), x(3)).$$

Suppose $\{x_n\}$ be a sequence of elements of M such that $Gx_{n+1} = (1-t)\,Gx_n + tFx_n$, for $n \geqslant 1$ and $t \in (0, 1)$.

Setting $t = 1/3$ and consider $x_1 = (1, 0, 0) \in M$, then it is easy to see that $x_2 = (2/3, -1/3, 0)$.

$x_3 = (5/9, -4/9, 0)$ etc.

Taking $x = x_1$, $y = x_2$ and $1/6 \leqslant q < 1$, we see that all the conditions of Theorem 1 are satisfied and **0** is the unique common fixed point of F and G.

Remark : Using (a_3) of Khan's condition in example 1 for $x = x_1$ and $y = x_2$, we have

$$\|Fx_1 - Fx_2\| \leqslant \alpha\,\|Gx_1 - Gx_2\|$$

i. e. $\sqrt{2}/3 \leqslant \alpha\,\sqrt{2}/3$

a contradiction, since $\alpha \in [0, 1)$ for uniqueness of common fixed point of maps F and G. Hence condition (a_3) fails to prove the existance of the unique common fixed point of maps F and G. But this pair of mappings F and G is contained in our class of mappings and hence it guarantees the uniqueness of common fixed point of maps F and G.

References

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