Indian J. pure appl. Math., 17 (8) : 974-993, August 1986

FIXED POINT THEOREMS IN 2-METRIC SPACES

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(*Received* 1 *November* 1985)

The notion of weak continuity of a 2-metric and the concept of weak commutativity of a pair of self-maps on a 2-metric space are introduced and they are exploited in obtaining certain fixed point theorems for a family of four self-maps on a 2-metric space, generalizing some of the known results.

In this paper we consider four self-maps on a 2-metric space satisfying a generalized contraction condition governed by a control function and explore the possibilities for them to admit a common fixed point and thereby obtain certain generalizations of some of the theorems of Khan and Fisher[3], Lal and Das[4], Rhoades[7]. Singh and Ram[9] and Singh, Tiwari and Gupta[10].

We also provide examples to show that a convergent sequence in a complete 2-metric space need not be Cauchy and that the 2-metric of a complete 2-metric space in which every convergent sequence is Cauchy need not be continuous.

SECTION 0

We give below the following definitions from Gähler[5] and White[11] for the sake of completeness.

Definition 0.1 – Let X be a non-empty set. A real valued function d on $X\times X\times X$ is said to be a 2-metric on X if

(i) to each pair of distinct points x, y in X, there exists a point z in X such that $d(x, y, z) \neq 0$,

(ii) $d(x, y, z) = 0$ when at least two of x, y, z are equal

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all x, y, z in X and

(iv) $d(x, y, z) \leqslant d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all x, y, z, w in X.

When d is a 2-metric on X, the ordered pair (X, d) is called a 2-metric space.

*Research supported by U. G. C., New Delhi.

Definition 0.2—A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent with limit x in X if

$\lim_{n\to\infty} d(x_n, x, a) = 0$ for all a in X.

Definition 0.3—A 2-metric on a set X is said to be continuous on X if it is sequentially continuous in two of its arguments.

It is known that a 2-metric is a non-negative real valued function, that it is continuous in any one of its arguments and that if it is continuous in two of its arguments then it is continuous in all the three arguments.

Definition 0.4—A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be a Cauchy sequence if

$$\lim_{m,n\to+\infty} d(x_n, x_m, a) = 0 \text{ for all } a \text{ in } X.$$

Definition 0.5—A 2-metric space is said to be complete if every Cauchy sequence in it is convergent.

Remark 0.1 : In a (complete) 2-metric space a convergent sequence need not be Cauchy. Example 0.1 illustrates it.

Example 0.1—Let $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.

Define $d : X \times X \times X \to [0, \infty)$ as

$$d(x, y, z) = \begin{cases} 1 \text{ if } x, y, z \text{ are distinct and } \left\{\frac{1}{n}, \frac{1}{n+1}\right\} \subset \{x, y, z\} \\ \qquad \text{for some positive integer } n \\ 0 \text{ otherwise.} \end{cases}$$

Then it can be verified that (X, d) is a complete 2-metric space. The sequence $\left\{\frac{1}{n}\right\}$ converges to zero.

But $\left\{\frac{1}{n}\right\}$ is not Cauchy.

Remark 0.2 : In a 2-metric space (X, d), every convergent sequence is Cauchy when d is continuous on X. Example (0.2) shows that the converse need not be true.

Example 0.2—Let $X = \{a\} \cup \{a_n \mid n = 1, 2, \ldots\} \cup \{b\} \cup \{b_n \mid = 1, 2, \ldots\}$,

where $a = (1, 0)$, $b = (0, 0)$, $a_n = \left(1 + \frac{1}{n}, 0\right)$ and $b_n = \left(0, \frac{1}{n}\right)$ $(n = 1, 2, \ldots)$.

Define $d : X \times X \times X \to [0, \infty)$ as

$$d(x, y, z) = \begin{cases} 1 \text{ if } \{x, y, z\} = \{a_n, b_n, a\} \text{ or } \{a_n, b_n, b\} \text{ for some } n \in N \\ \qquad \text{or } \{a_n, b_n, a_m\} \text{ or } \{a_n, b_n, b_m\} \text{ for some} \\ \qquad\qquad m, n \in N \text{ with } m \neq n, \\ \Delta\, x\, y\, z \text{ otherwise,} \end{cases}$$

where N is the set of all positive integers and Δxyz is the area of the triangle formed by the points x, y and z.

It can be verified that (X, d) is a complete 2-metric space and that every convergent sequence in it is Cauchy.

But d is not continuous on X, for $\{a_n\}$ converges to a, $\{b_n\}$ converges to b and $\{d(a_n, b_n, a)\}$ does not converge to zero.

In view of Remarks 0.1 and 0.2 we introduce the following

Definition 0.6—A 2-metric d on a set X is said to be weakly continuous on X if every convergent sequence in it is Cauchy.

Remark 0.3 : A 2-metric d on a set X is weakly continuous on X iff for every convergent sequence $\{x_n\}$ in X, the sequence $\{d(x_n, x_{n+1}, a)\}$ converges to zero for each a in X.

Remark 0.4 : If d is a weakly continuous 2-metric on X and $\{x_n\}$, $\{y_n\}$ are convergent sequences in X with the same limit then the sequence $\{d(x_n, y_n, a)\}$ converges to zero for each a in X.

Remark 0.5 : When d is a 2-metric on a finite set X then, but for the first few terms, all the terms of a convergent/Cauchy sequence in (X, d) are the same and hence (X, d) is complete, d is continuous on X and any self-map on X is continuous on X.

In analogy with the notion of weak commutativity in metric spaces[8] we introduce the concept of weak commutativity of two self-maps on a 2-metric space as follows.

Definition 0.7—We say that two self-maps S, T on a 2-metric space (X, d) are weakly commutative on X or commute weakly on X or (S, T) is a weakly commuting pair (w. c. p.) on X if

$$d(STx, TSx, a) \leqslant d(Tx, Sx, a)$$

for all x, a in X.

Remark 0.6 : A commuting pair of self-maps on a 2-metric space is weakly commutative. Example 0.3 shows that the converse need not be true.

Example 0.3—Let $X = \{1, 2, 3, 4\}$.

Define $d : X \times X \times X \to [0, \infty)$ as

$$d(x, y, z) = \begin{cases} 0 \text{ if } x = y \text{ or } y = z \text{ or } z = x \text{ or } \{x, y, z\} = \{1, 2, 3\} \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

Define $S, T : X \to X$ as $S1 = S2 = S3 = S4 = 2$ and

$$T1 = T2 = T3 = T4 = 3.$$

Then (X, d) is a 2-metric space and S, T commute weakly on X but they do not commute on X.

The mappings $f, g : X \to X$ defined by $f1 = f2 = 3$, $f3 = f4 = 2$, $g1 = g3 = 2$ $g2 = g4 = 3$ also commute weakly but do not commute.

SECTION 1

Throughout this section :

(X, d) is a 2-metric space ;

S, T are self-maps on X;

unless otherwise stated f, g are also self-maps on X;

R^+ is the set of all non-negative real numbers; and

$\phi : R^+ \to R^+$ is an increasing map with $\phi(t+) < t \ \forall \ t \in (0, \infty)$.

Lemma 1.1—Suppose that

$$d(fx, gy, a) \leqslant \phi(\max\{d(Sx, Ty, a), d(Sx, fx, a), d(Ty, gy, a), d(fx, Ty, a), d(Sx, gy, a)\}) \quad \ldots(I)$$

for all x, y, a in X and that there are sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\left.\begin{aligned} \text{and} \quad fx_{2n} &= Tx_{2n+1} = y_{2n} \\ gx_{2n+1} &= Sx_{2n+2} = y_{2n+1} \end{aligned}\right\} \quad \ldots(II)$$

for every non-negative integer n. Then $d(y_i, y_j, y_k) = 0$

for all non-negative integers i, j, k.

PROOF : Taking $x = x_{2n+2}$, $y = x_{2n+1}$ and $a = y_{2n}$ in inequality (I) we get

$$d(y_{2n}, y_{2n+1}, y_{2n+2}) \leqslant \phi(d(y_{2n}, y_{2n+1}, y_{2n+2})).$$

Taking $x = x_{2n+2}$, $y = x_{2n+3}$ and $a = y_{2n+1}$ in inequality (I) we obtain

$$d(y_{2n+1}, y_{2n+2}, y_{2n+3}) \leqslant \phi(d(y_{2n+1}, y_{2n+2}, y_{2n+3})).$$

Thus we have

$$d(y_n, y_{n+1}, y_{n+2}) \leqslant \phi(d(y_n, y_{n+1}, y_{n+2}))$$

for every non-negative integer n.

Since $\phi(t) < t \ \forall \ t \in (0, \infty)$ we must have

$$d(y_n, y_{n+1}, y_{n+2}) = 0 \quad \ldots(III)$$

for every non-negative integer n.

For $a \in X$, let $d_n(a) = d(y_n, y_{n+1}, a)$ $(n = 0, 1, 2, \ldots)$.

On using equation (III[1]), from (iv) we obtain

$$d(y_n, y_{n+2}, a) \leqslant d_n(a) + d_{n+1}(a) \qquad \ldots(IV)$$

for every non-negative integer n and for all a in X.

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in inequality (I) and using inequality (IV) we get

$$d_2(a) \leqslant \phi(d_{2n}(a) + d_{2n-1}(a))$$

for all a in X.

Taking $x = x_{2n}$ and $y = x_{2n+1}$ in inequality (I) and using inequality (IV) we get

$$d_{2r}(a) \leqslant \phi(d_{2n-1}(a) + d_{2n}(a))$$

for all a in X.

Thus we have

$$d_n(a) \leqslant \phi(d_{n-1}(a) + d_n(a)) \qquad \ldots(V)$$

for every positive integer n and for all a in X.

From inequality (V) it is clear that when $d_{n-1}(a)$ vanishes so does $d_n(a)$.

Noting that $d_0(y_0) = 0$, by a repeated application of inequality (V) we obtain

$$d_n(y_0) = 0$$

for every positive integer n.

For any positive integer m, noting that $d_{m-1}(y_m) = 0$, by a repeated application of inequality (V) we get

$$d_n(y_m) = 0 \ \forall\ n \geqslant m - 1.$$

For $0 \leqslant n < m - 1$, noting that $d_{m-1}(y_{n+1}) = 0 = d_{m-1}(y_n)$, from (iv) we obtain.

$$d_n(y_m) \leqslant d_n(y_{m-1}).$$

Noting that $d_n(y_{n+1}) = 0$, by a repeated application of the above inequality we get

$$d_n(y_m) = 0 \qquad (0 \leqslant n < m - 1).$$

Thus we have shown that

$$d_n(y_m) = 0 \ \forall\ n, m = 0, 1, 2, \ldots.$$

For non-negative integers i, j, k with $i < j$, noting that

$d_{j-1}(y_i) = 0 = d_{j-1}(y_k)$, from (iv) we get

$$d(y_i, y_j, y_k) \leqslant d(y_i, y_{j-1}, y_k).$$

By a repeated application of the above inequality we get

$$d(y_i, y_j, y_k) \leqslant d(y_i, y_i, y_k)$$

so that

$$d(y_i, y_j, y_k) = 0.$$

Thus

$$d(y_i, y_j, y_k) = 0 \ \forall \ i, j, k = 0, 1, 2, \ldots.$$

Note 1.1 : In Lemma 1.1 the condition '$\phi(t+) < t \ \forall \ t \in (0, \infty)$' may be replaced by the less stringent condition '$\phi(t) < t \ \forall \ t \in (0, \infty)$'.

Theorem 1.1—Suppose that inequality (I) holds for all x, y, a in X and that there are sequences $\{x_n\}$, $\{y_n\}$ in X such that eqns. (II) hold for every non-negative integer n and the sequence $\{d(y_n, y_{n+1}, a)\}$ converges to zero for each a in X. Then $\{y_n\}$ is Cauchy. Suppose that the sequence $\{y_n\}$ converges to some z in X. Then the following statements hold :

(1) f and S can have no common fixed point other than z.

(2) if $z \in S(X)$ and (f, S) is a w. c. p. then $Sz = z = fz$.

(3) If f and S are continuous at z and (f, S) is a w. c. p. then $Sz = z = fz$.

(4) If d is weakly continuous on X, S is continuous at z and (f, S) is a w. c. p. then $Sz = z = fz$.

(5) If d is weakly continuous on X, f is continuous at z and (f, S) is a w. c. p. then $fz = z$.

(6) If $fS = Sf$, f is continuous at z and $\{d(fy_n, fy_{n+1}, a)\}$ converges to zero for each a in X then $fz = z$.

(7) If $fS = Sf$, S^k is continuous at z for some positive integer k and $\{d(S^k y_n, S^k y_{n+1}, a)\}$ converges to zero for each a in X then $Sz = z = fz$.

(8) If d is weakly continuous on X, S^k is continuous at z for some positive integer k and $fS = Sf$ then $Sz = z = fz$.

(9) If $z \in S(X)$, $Sz = Tz$ and (f, S) is a w. c. p. then $fz = gz = Tz = Sz = z$.

(10) Statements 1 to 9 remain valid when f, S, g, T are replaced by g, T, f, S respectively.

(11) If $z \in S(X) \cap T(X)$ and (f, S), (g, T) are w. c. p.'s then z is a common fixed point of f, g, S and T.

(12) If f, g, S, T are continuous at z and (f, S), (g, T) are w. c. p.'s then z is a common fixed point of f, g, S and T.

(13) If d is weakly continuous on X; S, T are continuous at z and (f, S), (g, T) are w. c. p.'s then z is a common fixed point of f, g, S and T.

PROOF : Suppose $\{y_{2n}\}$ is not Cauchy.

Then $\exists\ c \in X$, $\epsilon > 0$ and strictly increasing sequences $\{m_k\}$, $\{n_k\}$ of positive integers such that $k \leqslant n_k < m_k$,

$$d(y_{2n_k}, y_{2m_k}, c) \geqslant \epsilon \text{ and } d(y_{2n_k}, y_{2m_k-2}, e) < \epsilon \qquad \ldots(VI)$$

for all $k = 1, 2, \ldots$.

In view of Lemma (1.1), from (iv) we have

$$d(y_{2n_k}, y_{2m_k}, c) - d(y_{2n_k}, y_{2m_k-2}, c)$$

$$\leqslant d(y_{2m_k-2}, y_{2m_k}, c)$$

$$\leqslant d_{2m_k-2}(c) + d_{2m_k-1}(c).$$

Since

$$\{d(y_{2n_k}, y_{2m_k}, c) - \epsilon\}_{k=1}^{\infty} \text{ and } \{\epsilon - d(y_{2n_k}, y_{2m_k-2}, c\}_{k=1}^{\infty}$$

are sequences of non-negative real numbers and $d_i(c) \to 0$ as $i \to \infty$, from the above inequality it follows that

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k}, c) = \epsilon \qquad \ldots(VII)$$

and

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-2}, c) = \epsilon.$$

From (iv) it is clear that

$$|d(x, y, a) - d(x, y, b)| \leqslant d(a, b, x) + d(a, b, y) \qquad \ldots(VIII)$$

for all x, y, a, b in X.

Taking $x = y_{2n_k}$, $y = c$, $a = y_{2m_k-1}$ and $b = y_{2m_k}$ in inequality (VIII) and noting that $d_{2m_k-1}(c) \to 0$ as $k \to \infty$, in view of Lemma 1.1 and eqn. (VII), we obtain

$$\lim_{k\to\infty} d(y_{2n_k}, y_{2m_k-1}, c) = \epsilon. \qquad \ldots(IX)$$

Noting that $d_{2n_k}(c) \to 0$ as $k \to \infty$, in view of Lemma 1.1 and equations (VII) and (IX), from inequality (VIII), we get

$$\lim_{k\to\infty} d(y_{2n_k+1}, y_{2m_k}, c) = \epsilon \qquad \ldots(X)$$

and

$$\lim_{k\to\infty} d\ y_{2n_k+1}, (y_{2m_k-1}, c) = \epsilon. \qquad \ldots(XI)$$

Taking $x = x_{2m_k}$, $y = x_{2n_k+1}$ and $a = c$ in inequality (I) and then taking limits on both sides as $k \to \infty$, on making use of equations (VII), (IX), (X) and (XI) and the fact that the sequences $\{d_{2m_k-1}(c)\}$ and $\{d_{2n_k}(c)\}$ converge to zero, we obtain

$$\epsilon \leqslant \phi(\epsilon +).$$

This is a contradiction since $\epsilon > 0$ and $\phi(t +) < t \ \forall\ t \in (0, \infty)$.

Hence $\{y_{2n}\}$ is Cauchy.

In view of Lemma 1.1, from (iv) we have

$$d(y_{2n}, y_{2m+1}, a) \leqslant d(y_{2n}, y_{2m}, a) + d_{2m}(a)$$

and

$$d(y_{2n+1}, y_{2m+1}, a) \leqslant d(y_{2n}, y_{2m+1}, a) + d_{2n}(a)$$

for all non-negative integers m and n and for all a in X.

Hence $\{y_n\}$ is also Cauchy.

Suppose $\{y_n\}$ converges to some $z \in X$.

(1) Suppose w is a common fixed point of f and S.

Taking $x = w$ and $y = x_{2n+1}$ in inequality (1) and then taking limits on both sides as $n \to \infty$ we obtain

$$d(w, z, a) \leqslant \phi(d(w, z, a) +) \ (a \in X).$$

Since $\phi(t +) < t \ \forall\ t \in (0, \infty)$ we must have $d(w, z, a) = 0$.

Since this is true for any a in X, from condition (I) of Definition (0.1) it follows that $w = z$.

(2) Suppose $z \in S(X)$.

Then there exists w in X such that $Sw = z$.

Taking $x = w$ and $y = x_{2n+1}$ in inequality (I) and then taking limits on both sides as $n \to \infty$ we obtain

$$d(fw, z, a) \leqslant \phi(d(fw, z, a) +)$$

for all a in X so that $fw = z$.

Suppose (f, S) is a w. c. p.,

Then $d(fz, Sz, a) = d(fSw, Sfw, a) \leqslant d(Sw, fw, a) = 0$ for all a in X

so that $fz = Sz$.

Taking $x = z$ and $y = x_{2n+1}$ in inequality (I) and then taking limlts on both sides as $n \to \infty$ and on making use of the fact that $Sz = fz$ we obtain

$$d(fz, z, a) \leqslant \phi(d(fz, z, a) +)$$

for all a in X so that $fz = z$. Hence $Sz = z$.

(3) Suppose S is continuous at z and (f, S) is a w. c. p..

Then $\{Sy_n\}$ converges to Sz and

$$\begin{aligned} d(fSx_{2n}, Sz, a) &\leqslant d(fSx_{2n}, Sfx_{2n}, a) + d(fSx_{2n}, Sfx_{2n}, Sz) \\ &\quad + d(Sfx_{2n}, Sz, a) \\ &\leqslant d(Sx_{2n}, fx_{2n}, a) + d(Sx_{2n}, fx_{2n}, Sz) \\ &\quad + d(Sy_{2n}, Sz, a) \\ &= d_{2n-1}(a) + d_{2n-1}(Sz) + d(Sy_{2n}, Sz, a) \end{aligned}$$

so that the sequence $\{d(fSx_{2n}, Sz, a)\}$ converges to zero for each a in X. Hence $\{fSx_{2n}\}$ converges to Sz.

Suppose now that f is continuous at z.

Then $\{fSx_{2n}\}$ converges to fz.

Hence $Sz = fz$.

As in the proof of statement 2 it can now be shown that

$$Sz = z = fz.$$

(4) Taking $x = Sx_{2n}$ and $y = x_{2m+1}$ in inequality (I) and then taking limits on both srdes as $m \to \infty$ we obtain

$$\begin{aligned} d(fSx_{2n}, z, a) \leqslant \phi(\max\{&d(SSx_{2n}, z, a), d(SSx_{2n}, fSx_{2n}, a) \\ &d(fSx_{2n}, z, a)\} +) \end{aligned} \qquad \text{...(XII)}$$

for every non-negative integer n and for all a in X.

Suppose S is continuous at z and (f, S) is a w. c. p..

Then $\{Sy_n\}$ converges to Sz and $\{fSx_{2n}\}$ converges to Sz.

Suppose now that d is weakly continuous on X.

Then $\{d(SSx_{2n}, fSx_{2n}, a)\}$ converges to zero for each a in X.

Now taking limits on both sides of inequality (XII) as $n \to \infty$ we obtain

$$d(Sz, z, a) \leqslant \phi(d(Sz, z, a) +)$$

for all a in X so that $Sz = z$.

From statement 2 it is now clear that $fz = z$.

(5) Taking $x = fx_{2n}$ and $y = x_{2m+1}$ in inequality (I) and then taking limits on both sides as $m \to \infty$ we obtain

$$d(ffx_{2n}, z, a) \leqslant \phi(\max\{d(Sfx_{2n}, z, a), d(Sfx_{2n}, ffx_{2n}, a), d(ffx_{2n}, z, a)\} +) \quad \ldots(XIII)$$

for any non-negative integer n and for any a in X.

Supppose f is continuous at z and (f, S) in a w. c. p..

Then $\{fy_n\}$ converges to fz and

$$\begin{aligned} d(Sfx_{2n}, fz, a) &\leqslant d(Sfx_{2n}, fSx_{2n}, a) + d(Sfx_{2n}, fSx_{2n}, fz) \\ &\quad + d(fSx_{2n}, fz, a) \\ &\leqslant d(fx_{2n}, Sx_{2n}, a) + d(fx_{2n}, Sx_{2n}, fz) \\ &\quad + d(fy_{2n-1}, fz, a) \\ &= d_{2n-1}(a) + d_{2n-1}(fz) + d(fy_{2n-1}, fz, a) \end{aligned}$$

so that the sequence $\{d(Sfx_{2n}, fz, a)\}$ converges to zero for each a in X. Hence $\{Sfx_{2n}\}$ converges to fz.

Suppose now that d is weakly continuous on X.

Then $\{d(Sfx_{2n}, ffx_{2n}, a)\}$ converges to zero for each a in X.

Now taking limits on both sides of inequality (XIII) as $n \to \infty$ we get

$$d(fz, z, a) \leqslant \phi(d(fz, z, a) +)$$

for all a in X so that $fz = z$.

(6) Suppose f is continuous at z and $fS = Sf$. Then $\{fy_n\}$ converges to fz and $Sfx_{2n} = fSx_{2n} = fy_{2n-1}$.

Taking $x = fx_{2n}$ and $y = x_{2m+1}$ in inequality (I) and then taking limits on both sides as $m \to \infty$ we obtain

$$d(fy_{2n}, z, a) \leqslant \phi(\max\{d(fy_{2n-1}, z, a), d(fy_{2n-1}, fy_{2n}, a), d(fy_{2n}, z, a)\} +)$$

for any non-negative integer n and for any a in X.

Suppose $\{d(fy_n, fy_{n+1}, a)\}$ converges to zero for each a in X.

Now taking limits on both sides of the above inequality as $n \to \infty$ we obtain

$$d(fz, z, a) \leqslant \phi(d(fz, z, a) +)$$

for all a in X so that $fz = z$.

(7) Suppose S^k is continuous at z for some positive integer k and $fS = Sf$. Then $\{S^k y_n\}$ converges to $S^k z$ and $fS^k x_{2n} = S^k f x_{2n} = S^k y_{2n}$.

Taking $x = S^k x_{2n}$ and $y = x_{2m+1}$ in inequality (I) and then taking limits on both sides as $m \to \infty$ we obtain

$$d(S^k y_{2n}, z, a) \leqslant \phi(\max\{d(S^k y_{2n-1}, z, a), d(S^k y_{2n-1}, S^k y_{2n}, a), d(S^k y_{2n}, z, a)\} +)$$

for any non-negative integer n and for any a in X.

Suppose $\{d(S^k y_n, S^k y_{n+1}, a)\}$ converges to zero for each a in X. Taking limits on both sides of the above inequality as $n \to \infty$ we get

$$d(S^k z, z, a) \leqslant \phi(d(S^k z, z, a) +)$$

for all a in X so that $S^k z = z$.

Hence $z \in S(X)$.

Now from statement 2 it follows that $fz = z$.

(8) Statement 8 is evident from statement 7.

(9) Suppose $z \in S(X)$ $Sz = Tz$ and (f, S) is a w. c. p..

From statement 2 it is clear that $fz = Sz = Tz = z$.

Taking $x = y = z$ in inequality (I) we obtain

$$d(z, gz, a) \leqslant \phi(d(z, gz, a))$$

for all a in X so that $gz = z$.

(10) Statement 10 can be established by proceeding along the lines of the proofs of statements 1 to 9 with obvious modifications.

Statements 11, 12 and 13 are evident from statements 2, 3, 4 and 10.

Remark 1.1 : Example 1.1 shows that in Theorem 1.1 one cannot conclude that the sequence $\{y_n\}$ is Cauchy if the condition '$\{d(y_n, y_{n+1}, a\}$ converges to zero for each a in X' is dropped even through X is finite (and $\phi(t) = \alpha t$, α being a constant in [0,1)).

Example 1.1—Let $X = \{1, 2, 3, 4, 5\}$.

Define $d: X \times X \times X \to R^+$ as

$$d(x, y, z) = \begin{cases} 0 \text{ if } x = y \text{ or } y = z \text{ or } z = z \text{ or } x, y, z \in \{1, 2, 3, 4\} \\ \frac{1}{2} \text{ if } \{x, y, z\} = \{1, 3, 5\} \text{ or } \{1, 4, 5\} \text{ or } \{2, 3, 5\} \text{or } \{2, 4, 5\} \\ 1 \text{ if } \{x, y, z\} = \{1, 2, 5\} \text{ or } \{3, 4, 5\}. \end{cases}$$

Define f, g, S, T from X into X as $f1 = f4 = 2$, $f2 = f3 = 1$, $f5 = 2$; $g1 = g3 = 4$, $g2 = g4 = 3$, $g5 = 4$; $S1 = 1$, $S2 = 2$, $S3 = 3$, $S4 = 4$, $S5 = 1$; and $T1 = 1$, $T2 = 2$, $T3 = 3$, $T4 = 4$ and $T5 = 3$.

Then d is a 2-metric on X, $fS = Sf$, $gT = Tg$ and

$$d(fx, gy, a) \leqslant \tfrac{1}{2} \max \{d(Sx, fx, a), d(Ty\, gy, a), d(fx, Ty, a), d(Sx, gy, a)\}$$

for all x, y, a in X.

For the sequences $\{x_n\}$, $\{y_n\}$ in X defined by

$$x_{4n} = 4, \quad x_{4n+1} = 2, \quad x_{4n+2} = 3, \quad x_{4n+2} = 1, \quad (n = 0, 1, 2, \ldots)$$

$$y_{4n} = 2, \quad y_{4n+1} = 3, \quad y_{4n+2} = 1, \quad y_{4n+3} = 4$$

Equations (I1) hold for every non-negative integer n.

But the sequence $\{y_n\}$ is not Cauchy, for the sequence $\{d(y_n, y_{n+1}, 5)\}$ does not converge to zero.

We note that neither f nor g has a fixed point.

By slightly strengthening inequality (I) the condition '$\{d(y_n, y_{n+1}, a)\}$ converges to zero for each a in X' can be dropped from Theorem (1.1). In fact, we have the following.

Theorem 1.2—Suppose that inequality (I) holds for all x, y, a in X and in addition

$$d(fx, gy, a) \leqslant \phi(\max \{d(Sx, Ty, a), d(Sx, fx, a), d(Ty, gy, a), \tfrac{1}{2}[d(fx, Ty, a) + d(Sx, gy, a)]\}) \qquad \ldots(\text{XIV})$$

whenever $a \in X$ and x, y in X are such that either $fx = Ty$ or $Sx = gy$. Suppose also that there are sequences $\{x_n\}$, $\{y_n\}$ in X such that eqns. (II) hold for every non-negative integer n. Then $\{y_n\}$ is Cauchy. Suppose that $\{y_n\}$ converges to some z in X. Then statements 1 to 13 of Theorem (1.1) hold here also. Further, the following statements are true.

(14) If $Sf = fS$, $Sg = gS$, $ST = TS$ and S^k is continuous at z for some positive integer k then $Sz = z = fz$.

(15) If $fS = Sf$, $fT = Tf$, $fg = gf$ and f is continuous at z then $fz = z$.

(16) If $fS = Sf$, $Sg = gS$, $ST = TS$, $fT = Tf$ and S^k is continuous at z for some positive integer k then z is a common fixed point of f, g, S and T.

(17) Statements 14, 15 and 16 remain valid when f, g, S, T are replaced by g, f, T, S respectively.

PROOF : Proceeding as in the proof of Lemma 1.1, in view of inequality (XIV), we get

$$d_n(a) \leqslant \phi(\max\{d_{n-1}(a), d_n(a), \tfrac{1}{2}[d_{n-1}(a) + d_n(a)]\})$$

for all a in X and for every positive integer n.

Since $\phi(t) < t \;\forall\; t \in (0, \infty)$, from the above inequality it is clear that

$$d_n(a) \leqslant \phi(d_{n-1}(a)) \qquad \ldots(XV)$$

for all a in X and for every positive integer n.

Hence $\{d_n(a)\}$ is a decreasing sequence.

Being a decreasing sequence of non-negative real numbers, the sequence $\{d_n(a)\}$ must converge to some non-negative real number $\alpha(a)$.

Now taking limits on both sides of inequality (XV) as $n \to \infty$ we obtain

$$\alpha(a) \leqslant \phi(\alpha(a) +)$$

for all a in X so that $\alpha(a) = 0$ for all a in X.

Thus $\{d_n(a)\}$ converges to zero for each a in X.

Now from Theorem (1.1) it is clear that $\{y_n\}$ is Cauchy.

Suppose $\{y_n\}$ converges to z for some z in X.

Now it is evident that statements 1 to 13 of Theorem 1.1 hold here also.

(14) Suppose $Sf = fS$, $Sg = gS$ and $ST = TS$.

From equations (II) it is now clear that

$$f(S^k x_{2n}) = T(S^k x_{2n+1}) = S^k y_{2n}$$

and

$$g(S^k x_{2n+1}) = S(S^k x_{2n+2}) = S^k y_{2n+1}$$

for every non-negative integer n.

In view of the above equations, it can be shown as in the proof of the convergence of the sequence $\{d_n(a)\}$ to zero that the sequence $\{d(S^k y_n, S^k y_{n+1}, a)\}$ also converges to zero for each a in X.

Statement 14 is now evident from statement 7.

(15) From the commutativity of f with each of the maps S, T and g it can be shown as in the proof of statement 14 that the sequence $\{d(fy_n, fy_{n+1}, a)\}$ converges to zero for each a in X.

Now statement 15 is evident from statement 6.

(16) Suppose $Sf = fS$, $Sg = gS$, $ST = TS$, $fT = Tf$ and S^k is continuous at z for some positive integer k.

From statement 14 we have $Sz = z = fz$.

Since $ST = TS$ and $fT = Tf$ it is now clear that Tz is a common fixed point of f and S.

From statement 1 we must have $Tz = z$.

From statement 9 it is now obvious that z is a common fixed point of f, g, S and T.

(17) Statement 17 can be established by proceeding along the lines of the proofs of statements 14 to 16 with suitable modifications.

Remark 1.2: If $\Psi : (R^+)^5 \to R^+$ is increasing in each coordinate variable and if $\zeta : R^+ \to R^+$ is defined by

$$\zeta(t) = \max\{\Psi(t, t, t, at, bt) \mid (a, b) \in \{(1, 1), (2, 0), (0.2)\}\} \quad \ldots(\text{XVI})$$

then ζ is an increasing map on R^+ and for all $t_1, t_2, t_3, t_4, t_5 \in R^+$,

$$\Psi(t_1, t_2, t_3, t_4, t_5) \leqslant \begin{cases} \zeta(\max\{t_1, t_2, t_3, t_4, t_5\}) \\ \zeta(\max\{t_1, t_2, t_3, \tfrac{1}{2}(t_4 + t_5)\}) \text{ if either} \\ t_4 = 0 \text{ or } t_5 = 0. \end{cases}$$

Conversely, if $\zeta : R^+ \to R^+$ is an increasing map on R^+ and if $\Psi : (R^+)_5 \to R^+$ is defined by

$$\Psi(t_1, t_2, t_3, t_4, t_5) = \begin{cases} \zeta(\max\{t_1, t_2, t_3, \tfrac{1}{2}(t_4 + t_5)\} \text{ if either} \\ \qquad t_4 = 0 \text{ or } t_5 = 0 \\ \zeta(\max\{t_1, t_2, t_3, t_4, t_5\}) \text{ otherwise} \end{cases}$$

then Ψ is increasing in each coordinate variable and equation (XVI) holds for all $t \in R^+$.

Remark 1.3: In view of Remark (1.2), the first sentence in the statement of Theorem (1.2) may be equivalently replaced by the following :

'Suppose that

$$d(fx, gy, a) \leqslant \Psi(d(Sx, Ty, a), d(Sx, fx, a), d(Ty, gy, a), d(fx, Ty, a), d(Sx, gy, a)) \quad \ldots(\text{XVII})$$

for all x, y, a in X, where $\Psi : (R^+)^5 \to R^+$ is increasing in each coordinate variable and $\zeta(t+) < t \; \forall \; t \in (0, \infty)$, where ζ is defined by eqn. (XVI)'.

Remark 1.4 : In view of Remark 1.3, Theorem 1 of Lal and Das[4] is a Corollary of Theorem 1.2.

Corollary 1.1—Suppose that (X, d) is complete and inequality (XIV) holds for all x, y, a in X. Suppose also that there are subsets Y_1, Y_2 of X such that $f(Y_1) \subseteq T(Y_2)$ and $g(Y_2) \subseteq S(Y_1)$. Then the sequence $\{y_n\}$ defined by eqns. (II) is Cauchy and converges to some z in X. Further, z is the unique common fixed point of f and S and also of g and T if one of the following statements holds :

1. f, g, S, T are continuous on X and (f, S), (g, T) are w. c. p.'s.
2. d is weakly continuous on X; S, T are continuous on X and (f, S), (g, T) are w. c. p.'s.
3. d is weakly continuous on X; S^k, T^l are continuous on X for some positive integers k and 1; and $fS = Sf$ and $gT = Tg$.
4. S^k is continuous on X for some positive integer k; $ST = TS, fS = Sf$, $gS = Sg$ and $fT = Tf$.
5. T^k is continuous on X for some positive integer k; $ST = TS, fT = Tf$, $gT_i = Tg$ and $gS = Sg$.
6. $Y_1 = Y_2 = X$; f is continuous on X; $fg = gf, fS = Sf, fT = Tf$ and (g, T) is a w. c. p.,
7. $Y_1 = Y_2 = X$; g is continuous on X; $fg = gf, gS = Sg$, $gT = Tg$ and (f, S) is a w. c. p.,
8. $Y_1 = Y_2 = X$; d is weakly continuous on X; (f, S), (g, T) are w. c. p.'s and either f or g is continuous on X.

PROOF : We prove that $fz = gz = Sz = Tz = z$ when statement 6 holds.

Suppose f is continuous on X, $fg = gf$, $fS = Sf$ and $fT = Tf$.

Then from statement 15 of Theorem 1.2 we have $fz = z$.

Suppose $Y_1 = X$.

Then $f(X) \subseteq T(X)$ and so $z \in T(X)$.

Suppose now that (g, T) is a w. c. p.

Then from statement 10 of Theorem 1.2 it is evident that

$$Tz = z = gz.$$

Suppose $Y_2 = X$.

Then $g(X) \subseteq S(X)$ and hence $z \in S(X)$.

Now from statement 2 of Theorem 1.2 we have $Sz = z = fz$.

From statements 1 and 10 of Theorem 1.2 it is clear that z is the unique common fixed point of f and S also of g and T.

Remark 1.5 : Theorem 4 of Rhoades[7], Theorem 1 of Singh, Tiwari and Gupta[16]. Theorem 2 (existence part of fixed point) of Khan and Fisher[3] and Theorem 1 of Singh and Ram[3] are all corollaries of Corollary 1.1. In fact, Corollary 1.1 shows that in the last two theorems the continuity condition on the 2-metric as well as that on some of the mappings under discussion is redundant.

Note 1.2 : Corollary 1.1 remains valid if the phrase 'inequality (XIV)' in it is replaced by the phrase 'inequality (XVII)', with the Ψ being as specified in Remark 1.3.

Note 1.3 : Metric space variants of Theorems 1.1 and 1.2 and Corollary 1.1 can be found in Naidu and Prasad[5].

Theorem 1.3—Let Y be a nonempty subset of X and f, g be mappings from Y into X. Suppose that inequality (I) holds for all x, y in Y and for all a in X, where $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$.

Suppose also that there are sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in Y such that

$$\left.\begin{aligned} &Sx_0 = Ty_0 = z_0 \\ \text{and} \quad &Sx_{n+1} = Ty_{n+1} = gy_n = fx_n = z_{n+1} \quad (n = 0, 1, 2, \ldots) \end{aligned}\right\}. \qquad \ldots(\text{XVIII})$$

Then $\{z_n\}$ is Cauchy. Suppose $\{z_n\}$ converges to some z in X. Then the following statements hold :

1. f and S can have no common fixed point other then z.
2. If $z \in Y \cap S(Y)$ and (f, S) is a w. c. p. than $Sz = z = fz$.
3. If $z \in Y$, (f, S) is a w. c. p. and f, S are continuous at z then $Sz = z = fz$.
4. If d is weakly continuous on X, S is continuous at z and (f, S) is a w. c. p. then $Sz = z$. If, further, $z \in Y$ then $fz = z$.
5. If $S(Y) \subseteq Y$, $fS = Sf$, S^k is continuous at z for some positive integer k and $\{d(S^k z_n, S^k z_{n+1}, a)\}$ converges to zero for each a in X then $S^k z = z$. If, further, $z \in Y$ then $Sz = z = fz$.
6. If d is weakly continuous on X, $S(Y) \subseteq Y$, $fS = Sf$ and S^k is continuous at z for some positive integer k then $S^k z = z$. If, further, $z \in Y$ then $Sz = z = fz$.
7. If $S(Y) \subseteq Y$, $fS = Sf$, $gS = Sg$ and $ST = TS$ on Y, and S^k is continuous at z for some positive integer k then $S^k z = z$. If, further $z \in Y$ then $Sz = z = fz$. If, in addition, $T(Y) \subseteq Y$ and $fT = Tf$ then $Tz = z = gz$.

8. If $z \in Y$, $fS = Sf$, f is continuous at z and $\{d(fz_n, fz_{n+1}, a)\}$ converges to zero for each a in X then $fz = z$.

9. If d is weakly continuous on X, $z \in Y$, f is continuous at z and (f, S) is a w. c. p. then $fz = z$.

10. If $z \in Y, f(Y) \subseteq Y, fS = Sf, fT = Tf$, $fg = gf$ and f is continuous at z then $fz = z$.

11. If $z \in Y \cap S(Y)$, $Sz = Tz$ and (f, S) is a w. c. p. then z is a common fixed point of f, g, S and T.

12. Statements 1 to 11 remain valid when f, g, S, T are replaced by g, f, T, S respectively.

PROOF : Taking $x = x_{i-1}$ and $y = y_{j-1}$ in inequality (I) and on making use of eqns. (XVIII) we get

$$d(z_i, z_j, a) \leqslant \phi\ (\max\{d(z_{i-1}, z_{j-1}, a), d(z_{i-1}, z_i, a), d(z_{j-1}, z_j, a), d(z_i, z_{j-1}, a), d(z_{i-1}, z_j, a)\}) \qquad \ldots(XIX)$$

for all positive integers i, j and for all a in X.

For $a \in X$, let

$$\beta_n(a) = \sup\{d(y_j, z_g, a)/0 \leqslant i, j \leqslant n\} \quad (n = 1, 2, \ldots).$$

For any positive integer n, from inequality (XIX) it is clear that

$$d(z_i, z_j, a) \leqslant \phi(\beta_n(a)) \ \forall\ i, j \in \{1, \ldots, n\} \text{ and } \forall\ a \in X. \qquad \ldots(XX)$$

Hence

$$\beta_n(z_0) \leqslant \phi(\beta_n(z_0)) \qquad (n = 1, 2, \ldots).$$

Since $\phi(t) < t\ \forall\ t \in (0, \infty)$ we must have $\beta_n(z_0) = 0\ \forall\ n = 1, 2, \ldots$.

Hence

$$d(z_p, z_q, z_0) = 0$$

for any non-negative integers p and q.

Hence from (iv) we have

$$d(z_0, z_j, a) \leqslant d(z_1, z_j, a) + d(z_0, z_1, a)$$

for any non-negative integer j and for any a in X.

Now from inequality (XX) we get

$$\beta_n(a) \leqslant \phi(\beta_n(a)) + d(z_0, z_1, a)$$

for any positive integer n and for any a in X.

Since $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$, from the above inequality it follows that $\{\beta_n(a)\}$ is bounded for each a in X.

Hence $\{d(z_i, z_j, a)/i, j > 0\}$ is bounded for each a in X.

For $a \in X$, let

$$\gamma_n(a) = \sup \{d(z_i, z_j, a)/i, j \geq n\} \ (n = 0, 1, 2, \ldots).$$

The sequence $\{\gamma_n(a)\}$, being a decreasing sequence of non-negative real numbers, converges to some non-negative real number $\gamma(a)$ From inequality (XIX) it is clear that

$$\gamma_n(a) \leq \phi(\gamma_{n-1}(a)) \ (n = 1, 2, \ldots)$$

for all a in X.

Taking limits on both sides of the above inequality as $n \to \infty$, we get

$$\gamma(a) \leq \phi(\gamma(a) +)$$

for all a in X.

Sinee $\phi(t+) < t \ \forall \ t \in (0, \infty)$ we must have $\gamma(a) = 0$ for all a in X.

Hence $\{z_n\}$ is Cauchy.

The proofs of statements 1 to 12 are analogous to those of the corresponding statements in Theorems 1.1 and 1.2.

Note 1.4 : In Theorem 1.3, the expressions $fS = Sf$ and (f, S) is a w. c. p. are respectively used in the sense that $fS = Sf$ on Y_1, and (f, S) is w. c. p. on Y_1, where $Y_1 = \{x \in Y \mid Sx \in Y\}$. Further, the expressions $gS = Sg$, $fg = gf$, (g, T) is a w. c. p. etc. are also used in a similar sense.

Corollary 1.2—Suppose that (X, d) is complete and

$$d(fx, fy, a) \leq \phi(\max \{d(x, y, a), d(x, fx, a), d(y, fy, a), d(fx, y, a), d(x, fy, a)\})$$

for all x, y, a in X, where $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$. Then f has a unique fixed point z and for any x in X the sequence $\{f^n x\}$ converges to z.

Remark 1.6 : Theorem 1 of Rhoades' is a special case of Corollary 1.2 with $\phi(t) = \alpha t \ \forall \ t \in R^+$, α being a constant in [0, 1).

Remark 1.7 : Example 1.2 shows that in Corollary 1.2 the function f may not have a fixed point (even when ϕ is identically zero on R^+) if the definition of 2-metric is altered by replacing condition (i) of Definition (0.1) by the weaker condition 'there

exist distinct elements a, b, c in X such that $d(a, b, c) \neq 0$' as was indeed done by some authors.

Example 1.2—Let $X = \{1, 2, 3, 4\}$.

Define $d : X \times X \times X \to R^+$ as

$$d(x, y, z) = \begin{cases} 0 \text{ if } x = y \text{ or } y = z \text{ or } z = x \text{ or } \{x, y, z\} = \{1, 2, 3\} \text{ or } \{1, 2, 4\} \\ 1 \text{ if } \{x, y, z\} = \{1, 3, 4\} \text{ or } \{2, 3, 4\}. \end{cases}$$

Define $f : X \to X$ as $f1 = 2$, $f2 = 1$, $f3 = 2$ and $f4 = 2$.

Then d satisfies conditions (ii), (iii) and (iv) of Definition (0.1) and $d(fx, fy, a) = 0 \; \forall \; x, y, a \in X$.

But f has no fixed point.

Corollary 1.3 —Suppose that (X, d) is complete and

$$d(fx, fy, a) \leqslant \phi(\max\{d(Sx, Sy, a), d(Sx, fx, a), d(Sy, fy, a), d(fx, Sy, a), d(Sx, fy, a)\}) \quad \ldots(XXI)$$

for all x, y, a in X, where $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$. Suppose also that there is a sequence $\{x_n\}$ in X such that

$$Sx_{n+1} = fx_n \; (n = 0, 1, 2, \ldots).$$

Then $\{fx_n\}$ is Cauchy and converges to some z in X. Further, the following statements hold :

1. f and S can have no common fixed point other than z.
2. If $z \in S(X)$ and (f, S) is a w. c. p. then $Sz = z = fz$.
3. If f and S are continuous at z and (f, S) is a w. c. p. then

$$Sz = z = fz.$$

4. If d is weakly continuous on X, S is continuous at z and (f, S) is a w. c. p. then $Sz = z = fz$.
5. If $fS = Sf$ and S^k is continuous at z for some positive integer k then $Sz = z = fz$.
6. If $fS = Sf$ and f is continuous at z then $fz = z$.
7. If d is weakly continuous on X, f is continuous at z and (f, s) is a w. c. p. then $fz = z$.

Remak 1.8 : Theorem 1 of Singh *et al.*[10] is a Corollary of Corollary 1.3.

Corollary 1.4 –Suppose that (X, d) is complete, S^k is continuous on X for some positive integer k and f is a mapping from $S^{k-1}(X)$ into $S^k(X)$. Suppose alsothat $fS = Sf$ on $S^{k-1}(X)$ and inequality (XXI) holds for all x, y in $S^{k-1}(X)$ and for all a in X, where $\lim_{t \to +\infty} [t - \phi(t)] = +\infty$.

Then f and S have a unique common fixed point

Note 1.5 : Metric space analogues of Theorem 1.3 and Corollaries 1.2, 1.3 and 1.4 can be found in Naidu and Prasad[6]. Corollaries 1.2 and 1.3 are (improvements over) the 2-metric space variants of the corresponding results (Theorems 2.1 and 4.1) of Das and Naik[1] in metric spaces.

ACKNOWLEDGEMENT

The authors wish to express their deep sense of gratitude to Dr D. R. K Sangameswara Rao and Dr K. P. R. Sastry for their invaluable help in the preparation of this paper. The authors also wish to express their heartfelt thanks to Dr Brian Fisher and Dr S. L. Singh for promptly responding to the authors' request for reprints of their research papers.

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