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# FIXED POINTS OF A PAIR OF MAPPINGS

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In this paper some fixed point theorems] for a pair of self-mappings have been proved and generalized the result of Dass and Gupta.

Chatterji[1] extended the result of Dass and Gupta[2] for pair of self-mappings. In this note we shall give another generalization of Dass and Gupta's result.

*Theorem* 1—Let $T_1$ and $T_2$ be two continuous self-mappings of a metric space $(X, d)$ such that

$$d(T_1 x, T_2 y) \leqslant \frac{d(x, T_1x)\, d(x, T_2y) + d(y, T_2y)\, d(y, T_1 x)}{d(x, T_2y) + d(y, T_1x)}$$

for all $x, y$ in $X$. If for some $x_0 \in X$, the sequence $\{x_n\}$ of elements $x_n$, where $x_{2n+1} = T_1 x_{2n}$, $x_{2(n+1)} = T_2 x_{2n+1}$, ... has a convergent subsequence $\{x_{n_k}\}$ converging to a point $\xi \in X$, then $\xi$ is a unique fixed point of $T_1$ and $T_2$.

PROOF : Since we have

$$d(x_{2n+1}, x_{2n+2}) = d(T_1 x_{2n}, T_2 x_{2n+1})$$

$$< \frac{d(x_{2n}, T_1x_{2n})\, d(x_{2n}, T_2x_{2n+1}) + d(x_{2n+1}, T_2x_{2n+1})\, d(x_{2n+1}, T_1x_{2n})}{d(x_{2n}, T_2x_{2n+1}) + d(x_{2n+1}, T_1x_{2n})}$$

$$= \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+2})\, d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}$$

$$= \frac{d(x_{2n}, x_{2n+1}).\, d(x_{2n}, x_{2n+2})}{d(x_{2n}, x_{2n+2})}.$$

Therefore

$$d(x_{2n+1}, x_{2n+2}) \leqslant d(x_{2n}, x_{2n+1}). \qquad \ldots(1)$$

Again

$$d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n}, T_2 x_{2n-1})$$

$$\leqslant \frac{d(x_{2n}, T_1 x_{2n}).\, d(x_{2n}, T_2x_{2n-1}) + d(x_{2n-1}, T_2x_{2n-1}).\, d(x_{2n-1}, T_1x_{2n})}{d(x_{2n}, T_2 x_{2n-1}) + d(x_{2n-1}, T_1 x_{2n})}$$

*(equation continued on p. 995)*

$$= \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n})\, d(x_{2n-1}, x_{2n+1})}{d(x_{2n}, x_{2n}) + d(x_{2n-1}, x_{2n+1})}$$

$$= \frac{d(x_{2n-1}, x_{2n})\, d(x_{2n-1}, x_{2n+1})}{d(x_{2n-1}, x_{2n+1})}$$

$$= d(x_{2n-1}, x_{2n})$$

$$\therefore d(x_{2n}, x_{2n+1}) \leqslant d(x_{2n-1}, x_{2n}). \qquad \ldots(2)$$

Therefore from (1) and (2) we get

$$d(x_{2n+1}, x_{2n+2}) \leqslant d(x_{2n}, x_{2n+1}) \leqslant d(x_{2n-1}, x_{2n}) \leqslant \ldots$$

Continuing in this way, we obtain that

$$d(x_{2n+1}, x_{2n+2}) \leqslant d(x_{2n}, x_{2n+1}) \leqslant \ldots \leqslant d(x_0, x_1).$$

So we have a monotonic sequence of positive real numbers. Let this monotonic sequence be converge to a real number $l$.

Since $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ in $X$ which converges to some $\xi \in X$, therefore

$$\lim \; x_{2n_k} = \xi.$$

Then we have to show that $\xi$ is a unique fixed point of $T_1$ and $T_2$.

Let if possible that $\xi \neq T_1 \xi$.

Now

$$d(\xi, T_1 \xi) = d(\lim x_{2n_k}, T_1 \lim x_{2n_k})$$

$$= \lim d(x_{2n_k}, T_1 x_{2n_k})$$

$$= \lim d(x_{2n_k}, x_{2n_k+1})$$

$$= \lim d(x_{2n_k+1}, x_{2n_k+2})$$

$$= \lim d(T_1 x_{2n_k}, T_2 T_1 x_{2n_k})$$

$$= d(\lim T_1 x_{2n_k}, T_2 \lim T_1 x_{2n_k})$$

$$= d(T_1 \xi, T_2 T_1 \xi). \qquad \ldots(3)$$

But

$$d(T_1 \xi, T_2 T_1 \xi) \leqslant \frac{d(\xi, T_1\xi)\, d(\xi, T_2T_1\xi) + d(T_1\xi, T_2T_1\xi)\, d(T_1\xi, T_1\xi)}{d(\xi, T_2T_1\xi) + d(T_1\xi, T_1\xi)}$$

*(equation continued on p. 996)*

$$= \frac{d(\xi, T_1\xi)\, d(\xi, T_2T_1\xi)}{d(\xi, T_2T_1\xi)}$$

$$= d(\xi, T_1\xi). \qquad \ldots(4)$$

Therefore from (3) and (4)

$$d(\xi, T_1\xi) = d(T_1\xi, T_2T_1\xi) \leqslant d(\xi, T_1\xi)$$

which is a contradiction

$$\therefore \xi = T_1\xi$$

that is $\xi$ is a fixed point of $T_1$.

In the same way let $\xi \neq T_2\xi$, again we get

$$d(\xi, T_2\xi) = d(T_1\xi, T_2T_1\xi) \leqslant d(\xi, T_2\xi)$$

a contradiction. Hence $\xi$ is a fixed point of $T_2$ also.

Therefore $\xi$ is a common fixed point of $T_1$ and $T_2$.

For the uniqueness of fixed point let $\eta \neq \xi$ be another common fixed point of $T_1$ and $T_2$. Then

$$d(\xi, \eta) = d(T_1\xi, T_2\eta)$$

$$< \frac{d(\xi, T_1\xi)\, d(\xi, T_2\eta) + d(\eta, T_2\eta)\, d(\eta, T_1\xi)}{d(\xi, T_2\eta) + d(\eta, T_1\xi)}$$

$$= 0$$

which is possible iff $\xi = \eta$. Therefore $\xi$ is a unique common fixed point of $T_1$ and $T_2$.

*Theorem* 2—Let $T_1$ and $T_2$ be two self-mappings of a metric space $(X, d)$ such that $T_1^p$ and $T_2^q$ are continuous and

$$d\left(T_1^p x, T_2^q y\right) < \frac{d\left(x, T_1^p x\right) d\left(x, T_2^q y\right) + d\left(y, T_2^q y\right) d\left(y, T_1^p x\right)}{d\left(x, T_2^q y\right) + d\left(y, T_1^p x\right)}$$

for all $x, y$ in $X$ and $p, q > 0$ are integers. If for some $x_0 \in X$, the sequence $\{x_n\}$ of elements $x_n$, where $x_{2n+1} = T_1^p x_{2n}, x_{2(n+1)} = T_2^q x_{2n+1}, \ldots$ has a convergent subsequence $\{x_{nk}\}$ converging to a point $\xi \in X$, then $\xi$ is a unique fixed point of $T_1$ and $T_2$.

PROOF : By Theorem 1, $T_1^p$ and $T_2^q$ have a unique fixed point in $X$.

Therefore $T_1^p \xi = \xi$ and $T_2^q \xi = \xi$.

Hence

$$T_1^p\ (T_1\xi) = T_1\left(T_1^p\ \xi\right) = T_1\xi$$

i. e. $T_1\xi$ is a fixed point of $T_1^p$ . But $\xi$ is a unique fixed point of $T_1^p$ and so $T_1\xi = \xi$.

Again

$$T_2^q\ (T_2\ \xi) = T_2\left(T_2^q\ \xi\right) = T_2\xi.$$

i. e. $T_2\xi$ is a fixed point of $T_2^q$ . Since $\xi$ is a unique fixed point of $T_2^q$ and so $T_2\xi = \xi$.

Therefore $\xi$ is a fixed point of $T_1$ and $T_2$.

For the uniqueness of fixed point let $\eta \neq \xi$ be another fixed point of $T_1$ and $T_2$ then

$$d(\xi, \eta) = d\left(T_1^p\ \xi,\ T_2^q\ \eta\right)$$

$$\leqslant \frac{d\left(\xi, T_1^p\ \xi\right) d\left(\xi, T_2^q\ \eta\right) + d\left(\eta, T_2^q\ \eta\right) d\left(\eta, T_1^p\ \xi\right)}{d\left(\xi, T_2^q\ \eta\right) + d\left(\eta, T_1^p\ \xi\right)}$$

$$= 0$$

which is possible iff $\xi = \eta$ and so $\xi$ is a unique fixed point of $T_1$ and $T_2$.

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