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ON CONCIRCULAR VECTOR FIELDS IN FINSLER SPACE

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In this paper we shall study the existence of concircular vector fields in a Finsler space. We investigate some special Finsler spaces under the assumption that they admit a concircular vector field.

1. INTRODUCTION

The concircular vector field in Riemannian manifold has been studied by Adati and Miyazawa[1]. A vector field X^i in Riemannian space is said to be concircular if $X^i_j = k_j X^i + \rho \delta^i_j$ where k_j is a gradient vector and ρ is a scalar function. The concurrent vector field is a particular case of concircular vector field in which k_j is a null vector and $\rho = -1$ (Sasaki[14] and Yano[16].) The concept of concurrent vector field in Finsler space has been introduced by Tachibana[15] and Matsumoto and Eguchi[8]. A vector field X^i on Finsler space is said to be concurrent if the following conditions are satisfied

$$\text{(a)}\quad X^i \mid_j = -\delta^i_j \qquad \text{(b)}\quad X^i \mid_j = 0 \qquad \ldots(1.1)$$

where $X^i \mid_j$ and $X^i \mid_j$ denote h-and v-covariant differentiations of X^i respectively (Matsumoto[4])

Generalizing the concept of concircular vector field of Riemannian manifold and concurreut vactor field of Finsler manifold we define the following

Definition—A vector field X^i in Finsler space F_n is said to be concircular if

$$\text{(a)}\quad X^i \mid_j = \frac{\partial X^i}{\partial x^i} - \frac{\partial X^i}{\partial y^i}\,\Gamma^{*i}_{hj}\, y^h + X^i\,\Gamma^{*i}_{ij} = k_j X^i + \rho\,\delta^i_j$$

$$\text{(b)}\quad X^i \mid_j = \frac{\partial X^i}{\partial y^j} + C^i_{jl}\, X^l = 0 \qquad \ldots(1.2)$$

where $\rho = \rho(x)$ and $k_i = \partial_j k(x)$.

2. The Identities Satisfied by a Concircular Vector Field

In terms of covariant components $X_i = g_{ij}X^j$ the relations (1.2a) and (1.2b) take the form

$$\text{(a)}\quad X_i \mid_j = \frac{\partial X_i}{\partial x^j} - \frac{\partial X_i}{\partial y^l}\,\Gamma^{*l}_{kj}\,y^k - X_l\,\Gamma^{*l}_{ij} = k_j X_i + \rho\, g_{ij}$$

$$\text{(b)}\quad X_i \mid_j = \frac{\partial X_i}{\partial y^j} - X_l\,C^l_{ij} = 0. \qquad \ldots(2.1)$$

These relations and Ricci identities for h-and v-covariant differentiations give the following identities :

$$R_{hijk}X^h = t_k g_{ij} - t_j g_{ik} \qquad \ldots(2.2)$$

$$P_{hijk}\,X^h = \rho C_{ijk} \qquad \ldots(2.3)$$

$$S_{hijk}\,X^h = 0 \qquad \ldots(2.4)$$

where R_{hijk}, P_{hijk} and S_{hijk} are the components of the Cartan's curvature tensors, $C_{ijk} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i\,\partial y^j\,\partial y^k}$, $\rho_k = \partial_k \rho$ and $t_k = \rho_k - \rho k_k$.

Since P_{hijk} and R_{hijk} are skew symmetric in h and i, from (2.2) and (2.3) we get

$$\text{(a)}\quad C_{ijk}X^i = = 0 \qquad \text{(b)}\quad t_k X_j - t_j X_k = 0. \qquad \ldots(2.5)$$

From (2.1b), (1.2b) and (2.5a) it follows that X^i and its covariant component X_i are functions of position only.

Theorem 2.1—The covariant component X_i and the contravariant component X^i of a concircular vector field of a Finsler space are functions of position only.

From (2.5b) it follows that there exist a scalar function λ such that

$$t_h = \lambda X_h \qquad \ldots(2.6)$$

and hence from (2.5a) we have

$$C^h_{ij}\,t_h = 0. \qquad \ldots(2.7)$$

The relation between Cartan's connection and Berwald's connection is given by[13]

$$G^h_{ij} = \Gamma^{*h}_{ij} + C^h_{ij}\mid_0. \qquad \ldots(2.8)$$

The relations (2.8), (1.2a), (2.5a) and Theorem 2.1 yields

$$X_{i(j)} = \frac{\partial X_i}{\partial x^j} - X_l\,G^l_{ij} = X_i \mid_j = k_j X_i + \rho g_{ij}, \qquad \ldots(2.9)$$

Hence we have the following:

Theorem 2.2—If a vector field X^i of a Finsler space is concircular with respect to the cartan's connection, it is concircular with respect to Berwald's connection.

A *c*-concircular vector field defined by Izumi[3] is the particular case of (1.2) in which $k_j = X_j$. Furthermore Izumi[3] has defined a concircular vector field X_i which satisfies the relation $X_{i(j)} = X_i X_j + \rho g_{ij}$. This relation is obtained by putting $k_j = X_j$ in (2.9).

The relation (2.9) and Ricci identity for Berwald's covariant differentiation give

$$H_{hijk} X^h = t_k g_{ij} - t_j g_{ik} \qquad ...(2.10)$$

where H_{hijk} is Berwald's curvature tensor. The relations (2.2) and (2.10) give

$$R_{hijk} X^h = H_{hijk} X^h. \qquad ...(2.11)$$

3. The Osculating Riemannian Metric Defined by a Concircular Vector Field of a Finsler Space

The osculating Riemannian metric $\bar{g}_{ij}(X)$ defined by concircular vector field X^i is is given by[13]

$$\bar{g}_{ij}(x) = g_{ij}(x, X) \qquad ...(3.1)$$

The Christoffel's symbols $\{^i_{jk}\}$ defined with respect to $\bar{g}_{ij}$ are given by

$$\{^i_{jk}\} = Y^i_{jk}(x, X) + C^i_{jr}(x, X)\ \partial_k X^r + C^i_{kr}(x, X)\ \partial_j X^r$$

$$- g^{ih}(x, X)\ C_{kjr}(x, X)\ \partial_h X^r.$$

Substituting the value of $\partial_k X^r$ from (1.2a) and using (2.5a) and Theorem (2.1) we get

$$\{^i_{jk}\} = \Gamma^{*i}_{jk}(x, X) + \rho\ C^i_{jk}(x, X). \qquad ...(3.2)$$

Contracting (3.2) by X^k we have in view of (2.5)

$$\{^i_{0k}\} X^k = \Gamma^{*i}_{jk}(x, X) X^k$$

which yield $X^i_{,j} = X^i \mid_j$. Hence we have the following :

Theorem 3.1—If a vector field X^i of a Finsler space is concircular, it is also concircular, with respect to the osculating Riemannian metric defined by X^i.

4. Some Special Finsler Spaces Admitting Concircular Vector Field

It is to be noted that a concurrent vector field X^i also satisfy the identity (2.5a) and Theorem 2.1. The proof of Theorems 2, 3, 4, 6 and 8 of Matsumoto and Eguchi[8]

are based on these two facts. Thus for a concircular vector field we have the following theorem.

Theorem 4.1—If a Finsler space admits a concircular vector field then each of the following spaces are Riemannian

(a) Landsberg space,

(b) 2-dimensional Finsler space,

(c) 3-dimensional Finsler space satisfying condition of Brickell's theorem

(d) C-reducible Finsler space,

(e) Finsler space of (α, β) metric.

Next we shall prove the following.

Theorem 4.2—If a Finsler space satisfying

$$L^2 C_{hij} \mid_k + C_{kij} y_h + C_{hkj} y_i + C_{hik} y_j + C_{hij} y_k = 0 \qquad \ldots(4.1)$$

admits a concircular vector field, the space is Riemannian.

PROOF : Contracting (4.1) by X^h and using (1.2b), and (2.5a) we get $C_{ijk} X^h y_h = 0$. If $X^h y_h = 0$ then $X^h \mid_k y_h = 0$ which in view of (1.2a) gives $\rho = 0$. But for a concircular vector field ρ is non-zero. Hence $C_{ijk} = 0$ which implies that the space is Riemannian.

Theorem 4.3—If a p-2 like Finsler space admits a concircular vector field with $\rho \neq X^h p_h$ then the space is Riemannian.

PROOF : A p-2 like Finsler space is characterized by the equation[5]

$$P_{hijk} = P_h C_{ijk} - P_i C_{hjk}. \qquad \ldots(4.2)$$

Contracting (4.2) with X^h and using (2.3) and (2.5a) we get $C_{ijk}(\rho - X^h P_h) = 0$. Hence $C_{ijk} = 0$ which implies that ths space is Riemannian.

Theorem 4.4—If a P-reducible Finsler space admits a concircular vector field, then the space is Riemannian.

PROOF : A P-reducible Finsler space F_n is characterized by the relation[9].

$$(n+1) P_{ijk} = (P_i h_{jk} + P_j h_{ki} + P_k h_{ij}) \qquad \ldots(4.3)$$

where

$$P_{ijk} = g_{ijk} \mid_0 = P_{hijk} y^h \text{ and } P_i = P_{ijk} g^{jk}.$$

The relation (2.3) gives $P_{ijk} X^i = 0$ and $P_i X^i = 0$. Contracting (4.3) with X^i we get

$$P_j h_{ki} X^i + P_k h_{ij} X^i = 0.$$

Transvecting this equation with P^j and using the fact that $h_{kl}X^k \neq 0$ we get $P_jP^j = 0$ which leads to $P_j = 0 = P_{ijk} = P_{hijk}$.

Hence inview of (2.3) we have $C_{ijk} = 0$ which implies that the space is Riemannian.

A Finsler space with the metric function $F(x, y) = [g_{ij}(y)\, y^i\, y^j]^{1/2} + \dfrac{\partial b}{\partial x^i} y^i$ (b is a scalar function of x^i) has a special form of P_{ijk} given by[11]

$$P_{ijk} = \lambda C_{ijk} + a_i\, h_{jk} + a_j\, h_{ik} + a_k\, h_{ij} \qquad \ldots(4.4)$$

where λ is a scalar function and a_i are components of a covariant vector. The properties of Finsler spaces having such form of P_{ijk} have been studied by the author[12]. Since $P_{ij0} = 0$, we have $a_0 = 0$. Contracting (4.4) with X^i and using eqns. (2.3) and (2.5a), the relation $P_{ijk} = P_{hijk}y^h$ and skew symmetric property of P_{hijk} in h, i, we get

$$a_iX^i\, h_{jk} + a_j\, h_{ik}\, X^j + a_k\, h_{ij}\, X^i = 0.$$

After contracting this equation with g^{jk} we get $a_i\, X^i = 0$ which gives

$$a_jh_{ik}\, X^i + a_k\, h_{ij}\, X^i = 0.$$

Transvecting this relation with a^j we get $a_ja^j = 0$ which gives $a_j = 0$. Hence inview of (4.4) we have $P_{ijk} = \lambda C_{ijk}$ i. e. F_n is P^* Finsler space.[2]

Theorem 4.5—If F_n is a Finsler space, satisfies (4.4) then F_n is P^* Finsler space.

A S3—like Finsler space is charaterized by the relation[5]

$$S_{hijk} = \frac{S}{(n-1)(n-2)}(h_{hj}\, h_{ik} - h_{hk}\, h_{ij}),\ n \geqslant 3 \qquad \ldots(4.5)$$

where S is v-scalar curvature given by $S = S_{hijk}\, g^{ik}\, g^{hj}$ and $h_{ij} = g_{ij} - L^{-2}\, y_i\, y_j$. Contracting (4.5) with X^h and using (2.4) we get

$$SX^h\,(h_{hj}\, h_{ik} - h_{hk}\, h_{ij}) = 0.$$

Transvecting this equation with g^{jk} we get $SX^h\ h_{hj} = 0$ which gives $S = 0$. This in view of (4.5). $S_{hijk} = 0$.

Theorem 4.6—If a $S3$-like finsler space admits a concircular vector field then the curvature tensor S_{hijk} vanishes.

Next we shall study the existence of concircular vector field in R-3 like Finsler space. An n dimensional ($n > 3$) Finsler space is said to be R-3 like[7] Finsler space if curvature tensor R_{hijk} is of the form

$$R_{hijk} = g_{hj}\, L_{ik} + g_{hk}L_{hj} - g_{ij}L_{hk} - g_{hk}L_{ij} \qquad \ldots(4.6)$$

where

$$L_{ik} = \left[R_{ik} - \frac{Rg_{ik}}{2(n-1)}\right] / n - 2 \qquad \ldots(4.7)$$

$$R_{ik} = g^{hj} R_{hijk}, \quad R = g^{ik} R_{ik}. \qquad ...(4.8)$$

The tensor L_{ik} satisfies the relation[8]

$$M_{hij} + L^{-2} (y_i M_{hj0} - y_j M_{hi0} + h_{hi} N_{j0} - h_{hj} N_{i0}) = 0 \qquad ...(4.9)$$

$$N_{ij} = L_{ij} - L_{ji} = L^{-2} (N_{i0} y_j - N_{j0} y_i) \qquad ...(4.10)$$

where

$$M_{hij} = C^r_{hi} L_{rj} - C^r_{hj} L_{ri}, \; L = L_{ij} g^{ij} = \frac{R}{2(n-1)}. \qquad ...(4.11)$$

Form (2.2), (2.6) and (4.6) it follows that

$$\lambda (X_k g_{ij} - X_j g_{ik}) = X_j L_{ik} - X_k L_{ij} + g_{ik} L_j - g_{ij} L_k \qquad ...(4.12)$$

where $L_j = L_{ij} X^i$. Contracting (4.12) with g^{ik} we get

$$(n-2) L_j = -[L + \lambda (n-1)] X_j.$$

Thus (4.12) will reduce to

$$L_{ij} - \frac{L+\lambda}{n-2} g_{ij} = X^{-2} \left[L_{ik} - \frac{L+\lambda}{n-2} g_{ik} \right] X^k X_j \qquad ...(4.13)$$

where X is the length of X^i. Hence (4.7) can be written as

$$R_{ij} - \left[\frac{R}{n-1} + \lambda \right] g_{ij}$$

$$= X^{-2} \left[R_{ik} - \left\{ \frac{R}{n-1} + \lambda \right\} g_{ik} \right] X^k X_j. \qquad ...(4.14)$$

Contracting (4.9) by X^h and using (2.5a), (4.11) we get $h_{hi} X^h N_{j0} - h_{hj} X^h N_{i0} = 0$. Since $h_{hi} X^h \neq 0$ there exists a scalar u such that $N_{i0} = u h_{hi} X^h$. On the other hand from (4.13) we get

$$N_{ij} = L_{ij} - L_{ji} = X^{-2} [L_{ik} X^k X_j - L_{jk} X^k X_i].$$

Contracting this with y^j we get

$$N_{i0} = X^{-2} [L_{ik} X^k X_0 - L_{0k} X^k X_i] = u (X_i - L^{-2} X_0 y_i).$$

Thus there exist scalars α, β such that

$$L_{ik} X^k = \alpha X_i + \beta y_i. \qquad ...(4.15)$$

Substituting (4.15) in (4.13) we get

$$L_{ij} = \frac{L+\lambda}{n-2} g_{ij} + X^{-2} \left[\alpha X_i + \beta y_i - \frac{L+\lambda}{n-2} X_i \right] X_j. \qquad ...(4.16)$$

In view of (4.11), (4.16) and (2.5a) we get $M_{hij} = 0$ so that (4.9) gives

$$h_{hi} N_{j0} - h_{hj} N_{i0} = 0.$$

Constructing this with g^{hi} we get $N_{j0} = 0$ so that (4.10) shows that L_{ij} is symmetric. Hence from (4.7) it follows that R_{ij} is symmetric and $R_{ik} X^k + \lambda (n-1) X_i = 0$. Thus (4.14) leads to

$$R_{ij} - \left[\frac{R}{n-1} + \lambda\right] g_{ij} = \left[\frac{R}{n-1} + n\right] X'_i X_j \qquad \ldots(4.17)$$

where $X'_i = \frac{X_i}{X}$ is a normalized concircular vector. This we have the following theorem

Theorem 4.7—If an R-3 like Finsler space with $R \neq 0$ admits a concircular vector field then the normalized concircular vector field X'_i is given by (4.17).

Corollary—Under the condition of Theorem 4.7

(i) The Ricci tensor R_{ij} is symmetric,

(ii) the matrix $\|R_{ij} - \left[\frac{R}{n-1} + \lambda\right] g_{ij}\|$ is of rank one.

A Finsler space is said to be an h isotropic if its curvature tensor R_{hijk} is given by[6]

$$R_{hijk} = R (g_{hj} g_{ik} - g_{hh} g_{ij}). \qquad \ldots(4.18)$$

Contracting this with X^h and using (2.2), (2.6) we get $R = -\lambda$. Thus (2.6) gives $t_k + R X_k = 0$. Hence we have the following:

Theorom 4.8—If an h-isotropic Finsler space admits a concircular vector field (1.2) then its scalar curvature R is given by

$$R X_k + t_k = 0.$$

Corollary—If an h-isotropic Finsler space admits a concircular vector field (1.2) with k_k as a gradient function of log ρ then the curvature tensor R_{hijk} vanishes.

A Finsler space of scalar curvature is characterized by the equation[8].

$$R_{hjk} = y^i R_{hijk} = \tfrac{1}{3} L^2 (R \mid_j h_{hk} - R \mid_k h_{hj}) + R(y_j h_{hk} - y_k h_{hj}) \qquad \ldots(4.19)$$

where R is a scalar curvature. Contracting (4.19) by X^h and using (2.2) we get

$$t_k y_j - t_j y_k = h_{hk} X^h (\tfrac{1}{3} L^2 R \mid_j + R y_j) - h_{hj} X^h (\tfrac{1}{3} L^2 R \mid_k + R y_k) \qquad \ldots(4.20)$$

If k_j is a gradient vector of log ρ then $t_j = 0$.

Contracting (4.20) by y^k and putting $t_j = 0$ we get $h_{hj} X^h R = 0$. Since $h_{hj} X^h \neq 0$, $R = 0$ which in view of (4.19) yield.

$$R_{hjk} = 0.$$

Hence $H_{ihjk} = \partial_i R_{hjk} = 0$.

The relation between curvature tensors R_{hijk} and H_{hijk} is given by[13]

$$R_{hijk} = \tfrac{1}{2}(H_{ijk} - H_{ihjk}) - C_{hjr} \mid_0 C^r_{ri} \mid_0 + C_{hkr} \mid_0 C^r_{ji} \mid_0 .$$

The vanishing of Berwald's curvature tensor H_{hijk} gives the following

$$R_{hijk} = C_{hkr} \mid_0 C^r_{ri} \mid_0 C^r_{ks} \mid_0 C_{hjr} \mid_0 \qquad \ldots(4.21)$$

Hence we have the following :

Theorem 4.9—If a Finsler space of scalar curvature admits a concircular vector field (1.2) with k_j as a gradient vector of $\log \rho$, the Berwald's curvature tensor vanishes and the Cartan's curvature tensor R_{hijk} is given by (4.21).

5. Recurrent Finsler Spaces Admitting Concircular Vector Field

A space is said to be a c^h-recurrent Finsler space if[6]

$$C_{hij} = \phi_l C_{hij} \ (\phi_l \neq 0). \qquad \ldots(5.1)$$

Contracting (5.1) by X^h and using (2.5) a we get $C_{hij} \mid_l X^h = 0$ which in view of (1.2) a and (2.5a) yield $C_{hij} = 0$. Hence we have the following :

Theorem 5.1—If a c^h-recurrent Finsler space admits a concircular vector field then it is Riemannian.

A P^h-recurrent Finsler space is characterised by the equation[10]

$$P_{jihk} \mid_l = \phi_l P_{jihk} \ (\phi_l \neq 0). \qquad \ldots(5.2)$$

The h-covariant differentiation of (2.3) gives in view of (1.2a)

$$P_{jihk} \mid_l X^j = -\rho P_{lihk} - (\rho k_l - \rho_l) C_{ihk} + \rho C_{ihk} \mid_l . \qquad \ldots(5.3)$$

Contracting (5.2) by X^j and using (2.3), (5.3) we get

$$\rho P_{lihk} = a_l C_{ihk} + \rho C_{ihk} \mid_l \qquad \ldots(5.4)$$

where

$$a_l = -\rho \phi_l - \rho k_l + \rho_l .$$

Contracting (5.4) with y^i we get

$$a_0 C_{ihk} = 0. \qquad \ldots(5.5)$$

Thus we have :

Theorem 5.2—If a P^h-recurrent Finsler space admits a concircular vector fields with $a_0 \neq 0$ then it is Riemannian.

A Finsler space F_n is said to be an R^h-recurrent space if the curvature tensor R_{jihk} satisfy the equation[10].

$$R_{jihk} \mid_l = \phi_l R_{jihk} \qquad ...(5.6)$$

where ϕ_l is a non-zero vector.

The h-covariant differentiation of (2.2) and the use of (1.2a) gives

$$R_{jihk} \mid_l X^j = (t_k \mid_l - k_l t_k) g_{ih} - (t_h \mid_l - k_l t_h) g_{ik} - \rho R_{lihk}. \qquad ...(5.7)$$

Contracting (5.6) with X^j and using (5.7), (2.2) we get

$$R_{lihk} = \phi_{kl} g_{ih} - \phi_{hl} g_{ik} \qquad ...(5.8)$$

where

$$\rho\phi_{kl} = - \phi_l t_k + t_k \mid_l - k_l t_k.$$

Since R_{lihk} is skew symmetric in l and i we have

$$\phi_{kl} g_{ih} + \phi_{ki} g_{lh} - \phi_{hl} g_{ik} - \phi_{hi} g_{lk} = 0$$

which by contraction with g^{ih} gives $\phi_{kl} = \frac{\phi}{n} g_{kl}$ where $\phi = \phi_{ij} g^{ij}$. Substituting the value of ϕ_{kl} in (5.8) we get

$$R_{lihk} = \frac{\phi}{n} (g_{ih} g^{kl} - g_{ik} g_{hl}).$$

Hence we have the following

Theorem 5.3—If a R^h-recurrent Finsler space admits a concircular vector field then it is h-isotropic with the scalar curvature $\frac{\phi}{n}$.

A S^h-recurrent Finsler space is characterized by the equation[10]

$$S_{jihk} \mid_l = \phi_l S_{jihk}, \ \phi_l \neq 0. \qquad ...(5.9)$$

Contracting (5.9) by X^j and using (2.4), (1.2a) we get $S_{lihk} = 0$. This gives the following :

Theorem 5.4—If a S^h-recurrent Finsler space admits a concircular vector field then the curvature tensor S_{lihk} vanishes.

Finally we consider the Finsler space whose Berwald curvature tensor is recurrent i. e.

$$H_{(ihk)(l)} = \phi_l H_{jihk}. \qquad ...(5.10)$$

A space for which (5.10) is satisfied will be called *H*-recurrent Finsler sdace. Contracting (5.10) with X^i and using (2.9), (2.10) we get

$$H_{ijhk} = \phi_{kj}\, g_{ih} - \phi_{hj}\, g_{ik} \qquad \ldots(5.11)$$

where we have used the fact that $t_k \mid_i = t_{k(i)}$.

Theorem 5.5—If a *H*-recurrent Finsler space admits a concircular vector field then the Berwald curvature tensor is given by (5.11).

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