Indian J. pure appl. Math., 17 (9) : 1092-1093, September 1986

NOTE ON A THEOREM OF ALTMAN

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(*Received* 6 *January* 1986)

The purpose of the present study is to clarify and Generalize a theorem of Altman[2].

Let Q be a real-valued function satisfying

(a) $0 < Q(s) < s$ for $s < 0$ and $Q(0) = 0$,

(b) $g(s) := s/(s - Q(s))$ is nonincreasing on $(0, \infty)$,

(c) $\int_0^{s_1} g(s)\, ds < \infty$ for each $s_1 > 0$, and

(d) $Q(s)$ is nondecreasing.

We suppose throughout that X is a complete metric space. A function $F : X \to X$ is called a generalized contraction if $d(Fx, Fy) \leqslant Q(d(x, y))$ for all $x, y \in X$. Altman[1] has shown that functions Q, having all four properties above, arise naturally in a Banach space setting.

The purpose of this note is to clarify and generalize the following theorem of Altman[2]:

Theorem 1—A generalized contraction has a unique fixed point z and, for every $x_0 \in X$, $\lim_{n\to\infty} (F)^n (x_0) = z$.

Actually, Altman's version has the following condition on Q replacing (a). (a^1) $0 < Q(s) < s$ for $0 < s \leqslant s_1$.

This form does not give uniqueness since s_1 depends on the starting point x_0. However, (a) is sufficient for uniqueness and the rest of the proof goes through as in Altman[2].

We generalize Theorem 1 for a pair of mappings, one of which is a generalized contraction and the other expansive.

Theorem 2—Let $F, G : X \to X$ be such that

(1) F is a generalized contraction,

(2) $d(Gx, Gy) \geqslant d(x, y)$ for all $x, y \in X$, and

(3) $F(X) \subseteq G(X)$.

Then $F(x) = G(x)$ has a unique solution z and, for every $x_0 \in X$,

$$\lim_{n\to\infty} (G^{-1}F)^n(x_0) = z.$$

PROOF : First, G^{-1} exists since (2) implies G is one-to-one. Also, (3) gives $G^{-1}F : X \to X$. Next it follows from (2) that

$$d(GG^{-1}x, GG^{-1}y) \geqslant d(G^{-1}x, G^{-1}y)$$

or

$$d(x, y) \geqslant d(G^{-1}x, G^{-1}y).$$

Replacing x by Fx and y by Fy in the latter inequality gives

$$d(G^{-1}Fx, G^{-1}Fy) \leqslant d(Fx, Fy) \leqslant Q(d(x, y)).$$

Hence $G^{-1}F$ is a generalized contraction with respect to Q. The result now follows by Theorem 1.

We note that taking G to be the identity map in Theorem 2 gives Altman's result.

In order to illustrate Theorem 2, we give the following example. Let $X = [0, \infty)$ and let d be the absolute value metric. Define $F, G : [0, \infty) \to [0, \infty)$ by $F(x) = \frac{1}{4+x}$ and $G(x) = 2x$. Let $Q(s) = \frac{s}{s+16}$. Clearly Q satisfies conditions (a), (b), (c) and (d). Then $d(Fx, Fy) = \frac{|x-y|}{xy + 4(x+y) + 16} \leqslant \frac{|x-y|}{|x-y| + 16}$ $= Q(d(x, y))$ and $d(Gx, Gy) = 2|x-y| \geqslant |x-y|$. Also $FX = \left(0, \frac{1}{4}\right] \subseteq [0, \infty) = GX$.

Hence, all conditions of Theorem 2 are fulfilled and $2x = \frac{1}{x+4}$ has a unique solution in $[0, \infty)$.

ACKNOWLEDGEMENT

We express our thanks to Professor V. M. Sehgal for his suggestions.

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