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SOME RESULTS IN THE FIXED POINT THEORY, III

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In this paper we give applications of our Localization Monotone Principle in the fixed point theory.

Let X be a topological space, $T : X \to X$ and let $B : X \to \mathbb{R}_+^0 := [0, \infty)$ be a T-orbitally lower semicontinuous function on X. A function B mapping X into the reals is T-orbitally lower semicontinuous at p if $\{x_n\}$ is a sequence in $O(x) := \{x, Tx, T^2 x, \ldots\}$, $x \in X$ and $x_n \to p$ implies that $B(p) \leqslant \liminf B(x_n)$. A space X is said to be T-orbitally complete if every Cauchy sequence which is contained in $O(x)$ for some $x \in X$ converges in X (c. f. Taskovic[14]).

In connection with this, in a recent paper[14] we introduced the concept of *TCS*-convergence in a space X; i. e., a topological space X satisfies the condition of *TCS*-convergence if there exists a point $x \in X$ such that $B(T^n x) \to 0$ $(n \to \infty)$ implies $\{T^n x\}$ has a convergent subsequence. We stated a new fixed point principle :

Localization Monotone Principle[14]—Let T be a mapping of a topological space X into itself, where X satisfies the condition of *TCS*-convergence. Suppose that there exists $y \in X$ and a mapping $\varphi : \mathbb{R}_+^0 \to \mathbb{R}_+^0$ such that $\varphi(t) < t$ and $\limsup_{z \to t+0} \varphi(z) < t$ for every $t > 0$ and

$$B(Tx) \leqq \varphi(B(x)), \text{ for every } x \in O(y) \qquad \ldots(1)$$

where $B : X \to \mathbf{R}$ is T-orbitally lower semicontinuous and $B(x) = 0$ implies $Tx = x$. Then T has a unique fixed point $\xi \in X$.

We get the following from the preceding principle :

Corollary 1—Let T be a mapping of a topological space X into itself. Suppose that there exist $\alpha \in [0, 1)$ and

$$B(Tx) \leqq \alpha B(x), \text{ for all } x \in X,$$

where $B : X \to \mathbf{R}_+^0$ is T-orbitally lower semicontinuous and $B(x) = 0$ implies $Tx = x$. If for some $x \in X$ the sequence $\{T^n x\}$ has a convergent subsequence, then T has a fixed point $\xi \in X$.

Corollary 2—Let T be a mapping of metric space (X, ρ) into itself and let X be T-orbitally complete. Suppose that there exist $\alpha \in [0, 1)$ and

$$A(Tx, Ty) \leqslant \alpha A(x, y), \text{ for all } x, y \in X$$

where $A : X \times X \to \mathbf{R}_+^0$, $x \to A(x, Tx)$ is lower semicontinuous and $\rho[x, y] \leqslant A(x, y)$ for all $x, y \in X$. Then T has a unique fixed point $\xi \in X$ and $T^n x \to \xi$ for each $x \in X$.

We notice, this result is generalization of the results of Chakrabarty[2].

In this paper, we apply our Localization Monotone Principle in the fixed point theory.

In this part we assume that (X, ρ) is a complete metric space and that Φ is a mapping of $\mathbf{R}_+^0$ into itself satisfying the conditions : Φ is nondecreasing and continuous on the right, $\Phi(t) < t$ for all $t > 0$. Further we assume that Ψ is a continuous mapping of $X \times X$ into $\mathbf{R}_+^0$ satisfying the conditions $\Psi(x, x) = 0$ and $\rho(x, y) \leqslant \Psi(x, y)$, for any x, y in X.

Corollary 3 (Chakrabarty[2])—Let M be a subset of X and T be a mapping of M into itself. Suppose that

$$\Psi(Tx, Ty) \leqslant \Phi(\Psi(x, y)), \text{ for all } x, y \in M \qquad \ldots(2)$$

where $\Psi(x, T^n x) \leqslant A(x)$, $n \in \mathbb{N}$ for every $x \in M$ and $A(x)$ a positive number. Then there is a point ξ in X such that for any $x \in X$, $T^n x \to \xi$ $(n \to \infty)$. If M is a closed, then ξ is the unique fixed point of T.

PROOF : Let $B(x) = \Psi(x, Tx)$ and $\varphi(t) = \Phi(t)$. It is easy to see that B and φ satisfy all the required hypotheses in localization monotone principle. Also, from (2) X satisfies the condition of TCS-convergence, because X is a complete metric space. This completes the proof.

Popa[9] obtained a fixed point theorem which is a common generalization of results of Banach, Ray[10] and Jaggi[7]. In this paper we extend this statement. Namely, we give applications of localization monotone principle, and as special case results of Popa[9], also and some others results.

Corollary 4 (Popa[9])—Let T be a continuous mapping of a Hausdorff spaces X into itself and let f be a continuous mapping of $X \times X$ into the nonnegative reals such that for some $a, b \in \mathbf{R}_+$, $a + b < 1$ and for all $x, y \in X$ and $x \neq y$:

$$f(Tx, Ty) \leqslant a f(x, Tx) f(y, Ty) (f(x, y))^{-1} + b f(x, y) \qquad \ldots(3)$$

and

$$f(x, x) f(y, y) \leqslant f^2(x, y) \text{ and } f(x, y) \neq 0. \qquad \ldots(4)$$

If for some $x \in X$ the sequence of iterates $\{T^n x\}$ has a convergent subsequence, then T has a unique fixed point.

PROOF: Let x be an arbitrary point in X. Then, for $y = Tx \neq x$, from (3) we have

$$f(Tx, T^2x) \leqslant (a + b) f(x, Tx), \text{ for all } x \in X.$$

Let $B(x) = f(x, Tx)$ which is lower semicontinuous on X, and let $\varphi(t) = (a+b)\, t$, for $t \in \mathbf{R}_+$; then B and φ satisfy all the required hypotheses in our localization monotone principle. Since X satisfies the condition of TCS-convergence (the sequence of iterates $\{T^n x\}$ has a convergent subsequence), applying our localization monotone principle we obtain $T\xi = \xi$ for some $\xi \in X$. Uniqueness follows immediately from conditions (3) and (4).

Corollary 5 (Jaggi[7], Bose and Mukherjee[8])—Let T be a mapping of a metric space X into itself satisfying the following condition

$$\rho[Tx, Ty] \leqslant a\, \rho[x, Tx]\, \rho[y, Ty]\, (\rho(x, y))^{-1} + b\, \rho[x, y], \; x \neq y$$

where a and b are nonnegative real numbers such that $a + b < 1$. If there exists an orbit $O(x_0)$ which contains a convergent subsequence of which T is orbitally continuous, then T has a unique fixed point.

PROOF: is analogous to the proof of the preceding statement of Popa[9].

We note, this statement of Bose and Mukherjee[1] is special case of Corollay 3 of Popa[9].

Let $\mathbf{N}$ denote the set of all nonnegative integers. A pair $(X, \to)$ of a set X and a subset $\to$ of the set $X^{\mathbf{N}} \times X$ is called an L-space if the following two conditions are satisfied:

(i) If $x_n = x \in X$ for all $n \in \mathbf{N}$, then $(\{x_n\}, x) \in \to$.

(ii) If $(s, x) \in \to$, then $(t, x) \in \to$ for every subsequence t of s.

In what follows, we shall write $s \to x$ or $x_n \to x$ instead of $(s, x) \in \to$, and read s converges to x, where $s = \{x_n\}$, $n \in \mathbf{N}$. If $s = \{x_n\}$, $n \in \mathbf{N}$, is a sequence in a set X, and if f is a mapping on X, then $f(s)$ denotes the sequence $\{f(x_n)\}$. Let $(X, \to)$ be an L-space. It is said to be separated if each sequence in X converges to at most one point of X. Let d be a nonnegative extended real valued function on $X \times X$. The L-space $(X, \to)$ is said to be d-complete if each sequence $\{x_n\}_{n \in \mathbf{N}}$ in X with $\sum_{n=0}^{\infty} d(x_{n-1}, x_n) < \infty$ converges to at least one point of X.

The following statement is corollary of our localization monotone principle:

Corollary 6 (Kasahara[6])—Let $(X, \to)$ be a separated L-space which is d-complete for a nonnegative extended real valued function d on $X \times X$, and f be a continuous mapping of X into itself satisfying the following conditions for some α, β with $0 \leqslant \alpha < 1$ and $0 < \beta \leqslant \infty$:

$$d(f(x), f^2(x)) \leqslant \alpha d(x, f(x)) < \alpha\beta, \text{ for all } x \in X. \qquad \ldots(5)$$

Then f has a fixed point, and the sequence $\{T^n x\}$ converges to a fixed point of f.

PROOF : As can readily be seen by induction, $d(f^n(a), f^{n+1}(a)) \leqslant \alpha^n d(a, f(a))$, for every $n \in \mathbb{N}$, where $\mathbb{N}$ denote the set of all nonnegative integers. Hence the d-completeness of the space implies that the sequence $\{f^n x\}$ converges to some $\xi \in X$. This implies that X satisfies the condition of TCS-convergence. Let $B(x) = d(x, f(x))$ and $\varphi(t) = \alpha t$. Applying localization monotone principle we get $\xi = f\xi$ for some $\xi \in X$.

Also, an immediate corollary of the our preceding localization monotone principle is :

Corollary 7—Let T be a mapping of metric space (X, ρ) into itself. Suppose that there exists $\alpha \in [0, 1)$ such that

$$\rho[Tx, T^2x] \leqslant \alpha\rho[x, Tx], \text{ for all } x \in X$$

where $x \to \rho[x, Tx]$ is T-orbitally lower semicontinuous and $\{T^n x\}$ has a convergent subsequence. Then T has a fixed point in X.

Corollary 8 (Dhage[3])—Let $T : X \to X$ be an orbitally continuous self-map of a metric space X and let X be T-orbitally complete. If T satisfies the condition

$$\min\{\rho[Tx, Ty], \rho[x, Tx], \rho[y, Ty]\}$$
$$+ a \min\{\rho[x, Ty], \rho[v, TX], \leqslant p\rho[x, y] + q\rho[x, Tx] \qquad \ldots(6)$$

for all $x, y \in X$ and a, p and q are real numbers such that $0 < p + q < 1$, then for each $x \in X$, then sequence $\{T^n x\}$ converges to a fixed point of T.

PROOF : Let $x \in X$ be an arbitrary point in X. Then, for $y = Tx$, from (6) we have

$$\min\{\rho[Tx, T^2x], \rho[x, Tx]\} \leqslant (p + q)\rho[x, Tx].$$

Hence, for $B(x) = \rho[x, Tx]$, $\varphi(t) = (p + q)t$, and since X satisfies the condition of TCS-convergence (X is T-orbitally complete metric space and $\rho[T^n x, T^{n+s}x] \leqslant (p+q)^n (1 - p - q)^{-1} \rho[x, Tx]$), applying our localization monotone principle we obtain $T\xi = \xi$ for some $\xi \in X$.

Some Remarks

(1) We notice, that and some other results (Theorems 2, 3 and 4) of Dhage[3], are special case of our localization monotone principle. Proofs are analogous of the preceding proof of Corollary 4.

(2) Also, in a paper[13] we have proved the following statement, which generalizes the preceding statement of Dhage[3]. A comparative study of these generalization has been made more recently by Yeh[15] and Khan[8].

Corollary 9 (Taskovic[13])—Let $T : X \to X$ be a mapping on X and let X be a T-orbitally complete metric space. If there exists real numbers α_i, β for every $x, y \in X$ such that $\alpha_1 + \alpha_2 + \alpha_3 > \beta$ and $\beta - \alpha_2 \geqq 0$ or $\beta - \alpha_3 \geqq 0$, and

$$\alpha_1 \rho [Tx, Ty] + \alpha_2 \rho [x, Tx] + \alpha_3 \rho [y, Ty] + \alpha_4 \min \{\rho [x, Ty], \rho [y, Tx]\} \leqslant \beta\rho [x, y] \quad ...(7)$$

then for each $x \in X$, then sequence $\{T^n x\}$ converges to a fixed point ξ of T.

Proof is analogous on the proof of the Corollary 6.

Also, we note, Corollary 6 of Dhage[3] which the assumption of the continuity of T is removed. Also, the condition (6) implies out the condition (7).

Let X denote a metric space and K a bounded subset of X. Following Kuratowski[5] we denote by $\alpha(K)$ the infimum of all $\epsilon > 0$ such that K admits a finite converging with subset of diameters less that ϵ. We use the following properties of the number α.

1. $\alpha(K) = 0$ if and only if K is precompact. For this reason $\alpha(K)$ is called the measure of noncompactness of K.

2. $\alpha(A \cup B) = \max \{\alpha(A), \alpha(B)\}$, $(A, B \subset X)$.

3. If K is compact then $\alpha(K) = 0$. Also, $\alpha(C|K) = 0 \Leftrightarrow \alpha(K) = 0$, and $0 \leqq \alpha(K) \leqq \delta(K)$.

4. $\alpha(K) = 0$ and X complete imply K is compact.

We list below some of contractive mappings for which various fixed point theorems have been stablished :

(1) (Furi and Vignoli[4]). The continuous mapping T is called densifying, if for every bounded subset A of X, such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$. Let F be a real lower semicontinuous function defined in $X \times X$. The densifying mapping T is said to be weakly F-contractive if the condition $F(Tx, Ty) < F(x, y)$ holds for all $x, y \in X$, $(x \neq y)$.

(2) (Ray and Chatterjee[11], Singh[12], Khan[8]). Let $F : X \times X \to \mathbf{R}_+^0$ be continuous and $T : X \to X$ be densifying mapping such that $F(Tx, Ty) < \alpha F(x, y) + \beta F(x, Tx) + \gamma F(y, Ty)$ for each pair, of distinct points $x, y \in X$ and for nonnegative real numbers α, β, γ with $\alpha + \beta + \gamma < 1$, then T is called generalized densifying.

(3) (Bose and Mukherjee[1]) Let F be a continuous symmetric mapping of $X \times X$ into the set of nonnegative reals, such that $F(x, y) = 0$ iff $x = y$ and

$$F(Tx, Ty) \leqslant aF(x, Tx)\, F(y, Ty)\, (F(x, y))^{-1} + bF(x, y),\ x \neq y$$

for each pair of distinct points $x, y \in X$ where $a + b < 1$ $(a, b \geqslant 0)$.

Corollary 10 (Furi and Vignoli[4])—Let $T : X \to X$ be a densifying (therefore, by definition, continuous) and weakly F-contractive mapping defined in a complete metric space. If for some $x_0 \in X$, the sequence $\{x_n\}$ of iterates starting from $x_0 = x$ is bounded, then T has a unique fixed point ξ in X.

PROOF : For $O(x)$ we have $TO(x) \subset O(x)$, then $O(x)$ is an invariant set. Denote by $\mathrm{Cl}O(x)$ the closure of $O(x)$. $\mathrm{Cl}O(x)$ is an invariant set, too; indeed, from the continuity of T it follows $T(\mathrm{Cl}\, O(x)) \subset \mathrm{Cl}\, T(O(x)) \subset \mathrm{Cl}\, O(x)$. Now we shall prove that $\mathrm{Cl}O(x)$ is compact. For this it is sufficient to show that $\alpha(O(x)) = 0$, since in a complete metric space the precompact sets are also relatively compact. Suppose $\alpha(O(x)) > 0$; in this case

$$\alpha(T(O(x))) < \alpha(O(x)) \text{ and since } O(x) = TO(x) \cup \{x\} \qquad \ldots(8)$$

it follows that

$$\alpha(O(x)) = \max\{\alpha(TO(x)), \alpha(x)\} + \max\{\alpha(TO(x)), 0\} = \alpha(TO(x)).$$

But this contradicts (8), hence $\alpha(O(x)) = 0$;

Also, let $B(x) = F(x, Tx)$. Since T is continuous and F is lower semicontinuous, B is lower semicontinuous, it is easy see to that B: $X \to \mathbf{R}_+^0$ satisfies all the required hypothesis in localization monotone principle. Hence, applying the localization monotone principle we obtain $T\xi = \xi$, for some $\xi \in X$. Uniqueness, it follows immediately from the weak F-contractivity.

Analogous to the proof for the preceding statement we have the same proof for the statements of Ray and Chatterjee[11], Singh[12], Khan[8], Bose and Mukherjee[1].

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