ON FEEBLY CLOSED MAPPINGS

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Recently the notion of feebly closed mappings was introduced. In this paper this concept is shown to coincide with the notion of $\alpha$-closed mappings. Furthermore it is shown that if the codomain is appropriately retopologised the concept of feebly closed mappings coincides with the usual notion of closed mappings. Some properties of $\alpha$-closed mappings are investigated.

1. INTRODUCTION

Let $S$ be a subset of topological space $(X, \mathcal{F})$. We denote the closure of $S$ and the interior of $S$ with respect to $\mathcal{F}$ by $\mathcal{F} \text{ cl } S$ and $\mathcal{F} \text{ int } S$ respectively, although we may suppress the $\mathcal{F}$ when there is no possibility of confusion. We denote the topology induced by $\mathcal{F}$ on $S$, by $\mathcal{F}_S$. Njastad$^8$ introduced the concept of an $\alpha$-set in $(X, \mathcal{F})$. A subset $S$ of $(X, \mathcal{F})$ is called an $\alpha$-set if $S \subseteq \mathcal{F} \text{ cl } (\mathcal{F} \text{ int } S)$. The notions of semi-open set and preopen set were introduced by Levine$^3$ and Mashhour et al.$^5$ respectively. A subset $S$ of $(X, \mathcal{F})$ is called a semi-open (respectively preopen) set if $S \subseteq \mathcal{F} \text{ cl } (\mathcal{F} \text{ int } S)$ (respectively $S \subseteq \mathcal{F} \text{ int } (\mathcal{F} \text{ cl } S)$). The complements of an $\alpha$-set, a semi-open set and a preopen set are called $\alpha$-closed, semi-closed and pre-closed respectively. Following Njastad$^8$ we denote the family of all $\alpha$-sets in $(X, \mathcal{F})$ by $\mathcal{F}^\alpha$. Njastad proved that $\mathcal{F}^\alpha$ is a topology on $X$. As any open set in $(X, \mathcal{F})$ is an $\alpha$-set, $\mathcal{F} \subseteq \mathcal{F}^\alpha$ in the lattice of topologies on the set $X$. If $A$ is a subset of $(X, \mathcal{F})$, then the intersection of all semiclosed sets containing $A$ is called the semi closure of $A$, and is denoted $s \text{ cl } A$. The largest semi-open set contained in $A$ is denoted by $s \text{ int } A$. Maheshwari and Tapi$^7$ defined $A$ to be a feebly open set in $(X, \mathcal{F})$ if there is an open set $U$ such that $U \subseteq A \subseteq s \text{ cl } U$. The complement of a feebly open set is called a feebly closed set.

2. FEEBLY CLOSED MAPS

The concepts of $\alpha$-closed and feebly closed mappings have been introduced by Mashhour et al.$^5$ and Maheshwari and Jain respectively.

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Definition 1—A function \( f : (X, \mathcal{F}) \to (Y, \mathcal{G}) \) is called

(i) feebly closed if the image of each closed set in \( X \) is feebly closed in \( Y \);

(ii) \( \alpha \)-closed if the image of each closed set in \( X \) is \( \alpha \)-closed in \( Y \).

Lemma 1—Let \( A \) be a subset of \( (X, \mathcal{F}) \). Then \( \text{s int} (\text{cl} A) = \text{cl} (\text{int} (\text{cl} A)) \).

Proof: Notice that \( \text{cl} (\text{int} (\text{cl} A)) \) is a semi-open set since \( \text{cl} (\text{int} (\text{cl} A)) = \text{cl} (\text{int} (\text{cl} (\text{int} (\text{cl} A)))) \), and \( \text{cl} (\text{int} (\text{cl} A)) \subset \text{cl} A \). Therefore \( \text{cl} (\text{int} \text{Cl} A) \subset \text{s int} \text{cl} A \).

Conversely, if \( U \) is any semi-open set contained in \( \text{cl} A \), then \( U \subset \text{cl} (\text{int} U) \subset \text{cl} (\text{int} (\text{cl} A)) \) and therefore \( \text{s int} (\text{cl} A) \subset \text{cl} (\text{int} (\text{cl} A)) \).

Maheshwari and Jain\(^6\) in Lemma 3 of their paper showed that a subset \( A \) of \( (X, \mathcal{F}) \) is feebly closed if and only if \( \text{s int} (\text{cl} A) \subset A \).

Proposition 1—If \( A \) is a subset of \( (X, \mathcal{F}) \), then \( A \) is feebly closed if and only if \( A \) is \( \alpha \)-closed.

Proof: It follows from the definitions of an \( \alpha \)-set and an \( \alpha \)-closed set that a subset \( A \) of \( (X, \mathcal{F}) \) is \( \alpha \)-closed if and only if \( \text{cl} (\text{int} (\text{cl} A)) \subset A \).

Since \( \text{cl} (\text{int} (\text{cl} A)) \subset A \) if and only if \( \text{s int} (\text{cl} A) \subset A \), by Lemma 1, \( A \) is \( \alpha \)-closed if and only if \( A \) is feebly closed.

An alternative proof of this result has been given in Proposition 1 of Janković and Reilly\(^3\), who proved that if \( A \) is a subset of \( (X, \mathcal{F}) \), then \( A \) is feebly open if and only if \( A \) is an \( \alpha \)-set.

Proposition 2 follows immediately from Proposition 1 and Definition 1.

Proposition 2—The following are equivalent:

1. \( f : (X, \mathcal{F}) \to (Y, \mathcal{G}) \) is feebly closed;
2. \( f : (X, \mathcal{F}) \to (Y, \mathcal{G}) \) is \( \alpha \)-closed;
3. \( f : (X, \mathcal{F}) \to (Y, \mathcal{G}^*) \) is closed.

If the codomain space of a feebly closed mapping \( f \) is retopologised in an obvious way, then \( f \) is simply a closed mapping. This observation puts the notion of feebly closed mappings into a more natural setting, and enables us to provide immediate proofs of some of the results in Maheshwari and Jain\(^6\). For example, Propositions 5 and 6 of Maheshwari and Jain\(^6\) are well-known results (Murdeswar\(^4\), Theorem 4.26, P. 96 and Theorem 4.28, p. 26) restated in this setting.

3. \( \alpha \)-Closed Mappings

The following classes of generalized closed mappings were introduced in Mashhour et al.\(^5\) and Noiri\(^9\).
Definition 2—A function \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) is called

(i) semi-closed if the image of each closed set in \( X \) is semi-closed in \( Y \);

(ii) preclosed if the image of each closed set in \( X \) is preclosed in \( Y \).

It is shown in Theorem 3 of Reilly and Vamanamurthy\(^{10}\) that a subset of \((X, \mathcal{I})\) is an \( \alpha \)-set if and only if it is semi-open and preopen. Thus we have the following result.

Proposition 3—A function \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) is \( \alpha \)-closed if and only if it is semi-closed and preclosed.

Examples 1 and 2 show that the separate converses are not in general true.

Example 1—Let \( X = \{a, b, c\} \) and define the topologies \( \mathcal{T} \) to be the discrete topology and \( \mathcal{U} = \{\phi, x, \{a\}, \{c\}, \{a, c\}\} \). We define \( f : (X, \mathcal{T}) \rightarrow (X, \mathcal{U}) \) by \( f(a) = f(b) = f(c) = a \). Then \( f \) is preclosed but not \( \alpha \)-closed since \( \{a\} \) is preclosed in \((X, \mathcal{U})\) but not \( \alpha \)-closed in \((X, \mathcal{U})\).

Example 2—If \( \mathcal{T} \) is the discrete topology and \( \mathcal{U} \) is the indiscrete topology then \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) is semi-closed but not \( \alpha \)-closed.

Andrijevic\(^1\) showed that if \( M \) is a subset of \((X, \mathcal{T})\) then \((\mathcal{T} M)^{\alpha} \subset (\mathcal{T}^{\alpha})_M\) (his Theorem 3.2) and if \( M \) is preopen then \((\mathcal{T} M)^{\alpha} = (\mathcal{T}^{\alpha})_M\) (his Theorem 3.6).

Proposition 4—Let \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) and \( f(X) \subseteq Y_1 \subseteq Y \).

1. If \( f : (X, \mathcal{T}) \rightarrow (Y_1, \mathcal{U}_1) \) is \( \alpha \)-closed, then so is \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \).

2. If \( Y_1 \) is preopen and \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) is \( \alpha \)-closed, then \( f : (X, \mathcal{T}) \rightarrow (Y_1, \mathcal{U}_{Y_1}) \) is \( \alpha \)-closed.

Proof: (1) If \( f : (X, \mathcal{T}) \rightarrow (Y_1, \mathcal{U}_{Y_1}) \) is \( \alpha \)-closed then \( f : (X, \mathcal{T}) \rightarrow (Y_1, (\mathcal{U}_{Y_1})^{\alpha}) \) is closed. By Andrijevic’s result \( f : (X, \mathcal{T}) \rightarrow (Y_1, (\mathcal{U}^{\alpha})_{Y_1}) \) is closed and by Theorem 4.24 (2) of Murdeshwar\(^4\) \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}^{\alpha}) \) is closed. Hence \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) is \( \alpha \)-closed.

(2) If \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \) is \( \alpha \)-closed then \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}^{\alpha}) \) is closed and by Theorem 4.24, of Murdeshwar, \( f : (X, \mathcal{T}) \rightarrow (Y_1, (\mathcal{U}^{\alpha})_{Y_1}) \) is closed.

If \( Y_1 \) is preopen then by Andrijevic’s result \( f : (X, \mathcal{T}) \rightarrow (Y_1, (\mathcal{U}_{Y_1})^{\alpha}) \) is closed, and so \( f : (X, \mathcal{T}) \rightarrow (Y_1, \mathcal{U}_{Y_1}) \) is \( \alpha \)-closed.

Proposition 5—If \( f \) is an \( \alpha \)-closed mapping \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}) \), \( \mathcal{T} \subseteq \mathcal{I} \) and \( \mathcal{U}^{\alpha} \subseteq \mathcal{U}_{1}^{\alpha} \) then \( f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{U}_{1}) \) is \( \alpha \)-closed.
Proof: If \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}) \) is \( \alpha \)-closed then \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}^*) \) is closed, and by Theorem 4.23 of Murdeshwar, \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}^*) \) is closed. Hence \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}_1) \) is \( \alpha \)-closed.

Proposition 6—If \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}) \) is an \( \alpha \)-closed mapping and \( B, C \subset Y \), then if \( f^{-1}(B) \) and \( f^{-1}(C) \) have disjoint neighbourhoods, \( B \) and \( C \) have disjoint neighbourhoods in \((Y, \mathcal{U}_1)\).

Proof: If \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}) \) is \( \alpha \)-closed then \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}^*) \) is closed and the result follows from Murdeshwar Theorem 4.28 (2).

Proposition 7—Let \( f : X \to (Y, \mathcal{U}) \) and let \( X \) be given the preimage topology \( \mathcal{F} \). Then \( f \) is \( \alpha \)-closed if and only if \( f(X) \) is \( \alpha \)-closed in \((Y, \mathcal{U}_1)\).

Proof: One implication, namely if \( f \) is \( \alpha \)-closed then \( f(X) \) is \( \alpha \)-closed in \((Y, \mathcal{U}_1)\), is clear since \( X \) is closed in the preimage topology.

Conversely, if \( f(X) \) is \( \alpha \)-closed in \((Y, \mathcal{U}_1)\) then \( f(X) \) is closed in \((Y, \mathcal{U}^*)\). Let \( \mathcal{F}_\alpha \) be a preimage topology on \( X \) for \( f : X \to (Y, \mathcal{U}^*) \).

Then, by Theorem 4.30 of Murdeshwar \( f : (X, \mathcal{F}_\alpha) \to (Y, \mathcal{U}^*) \) is closed. Since \( \mathcal{U} \subset \mathcal{U}^* \) and therefore \( \mathcal{F} \subset \mathcal{F}_\alpha \), \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}^*) \) is closed, and so \( f : (X, \mathcal{F}) \to (Y, \mathcal{U}) \) is \( \alpha \)-closed.

It is well known that the \( T_1 \) property is preserved under closed mappings. The following example shows that the \( T_1 \) property is not preserved under \( \alpha \)-closed mappings.

Example 3—Let \( X \) be an infinite set and \( p \) be a fixed point of \( X \). We define a topology \( \mathcal{F} \) on \( X \) as follows: for \( G \subset X \), \( G \in \mathcal{F} \) if \( G = \phi \) or \( G = X \) or \( X - G \) is finite. We define a topology \( \mathcal{U} \) on \( X \) as follows: for each \( G \subset X \), \( G \in \mathcal{U} \) if (i) \( G = \phi \) or \( G = X \), or (ii) \( G \subset X - \{p\} \) and \( X - G \) is finite. \((X, \mathcal{F})\) is \( T_1 \) but \((X, \mathcal{U})\) is not \( T_1 \) since for any point \( x \) distinct from \( p \) the only open set containing \( p \) namely \( X \), contains \( x \). Let \( f : (X, \mathcal{F}) \to (X, \mathcal{U}) \) be the identity function. Then \( f \) is \( \alpha \)-closed since if \( A \) is a closed subset of \((X, \mathcal{F})\) then either \( A \) is a closed subset of \((X, \mathcal{U})\) and therefore \( \alpha \)-closed in \((X, \mathcal{U})\), or \( A \) is finite, nonempty and \( p \notin A \). In this case \( \mathcal{U} \cl (\mathcal{U} \int (\mathcal{U} \cl A)) = \phi \subset A \) and therefore \( A \) is an \( \alpha \)-closed subset of \((X, \mathcal{U})\).

The following proposition is a generalization of the well known result that regularity is preserved under continuous, open and closed surjections (Murdeshwar Theorem 12.14, p. 206).

Lemma 2—If \( U \) and \( V \) are subsets of \((X, \mathcal{F})\), \( U \) is open and \( U \subset V \), then \( \cl U \subset \cl (\int (\cl V)) \).

Proof: If \( U \) is open and \( U \subset V \), then \( U \subset \cl V \) so that \( U \subset \int (\cl V) \). Therefore \( \cl U \subset \cl (\int (\cl V)) \).
Proposition 8—If \( f : (X, \mathcal{F}) \rightarrow (Y, \mathcal{U}) \) is an open, continuous, \( \alpha \)-closed surjection and \((X, \mathcal{F})\) is regular, then \((Y, \mathcal{U})\) is regular.

Proof: Let \( p \in Y \) and \( U \) be an open set in \((Y, \mathcal{U})\) containing \( p \). Let \( x \in X \) such that \( f(x) = p \). Since \((X, \mathcal{F})\) is regular there is an open set \( V \) in \((X, \mathcal{F})\) such that \( x \in V \subseteq \mathcal{F} \cap V \subseteq f^{-1}(U) \) so that \( p \in f(V) \subseteq f(\mathcal{F} \cap V) \subseteq U \). Since \( f \) is \( \alpha \)-closed, \( f(\mathcal{F} \cap V) \) is \( \alpha \)-closed and since \( f \) is open \( f(V) \) is open so that, by Lemma 2, \( \mathcal{U} \cap \text{cl} f(V) \subseteq \mathcal{U} \cap (\mathcal{U} \cap (\mathcal{U} \setminus \text{int} (\mathcal{U} \cap \text{cl} f(V)))) \subseteq U \) and therefore \( p \in f(V) \subseteq \mathcal{U} \cap \text{cl} f(V) \subseteq U \).

It is well known that normality is preserved under closed, continuous surjections Murdeshwar\(^4\) Theorem 15.3 (i). The following proposition is a generalization of this result.

Lemma 3—If \( U \) and \( V \) are subsets of \((X, \mathcal{F})\) and \( U \cap V = \emptyset \) then \( \text{int} (\text{cl} (\text{int} U)) \cap \text{int} (\text{cl} (\text{int} V)) = \emptyset \).

Proof: If \( U \cap V = \emptyset \) then \( \text{int} U \cap \text{int} V = \emptyset \), so that \( \text{int} U \cap \text{cl} (\text{int} V) = \emptyset \). Therefore \( \text{int} U \cap (\text{cl} (\text{int} V)) = \emptyset \) which implies that \( \text{cl} (\text{int} U) \cap \text{int} (\text{cl} (\text{int} V)) = \emptyset \), so that we have \( \text{int} (\text{cl} (\text{int} U)) \cap \text{int} (\text{cl} (\text{int} V)) = \emptyset \).

Proposition 9—If \( f : (X, \mathcal{F}) \rightarrow (Y, \mathcal{U}) \) is a continuous, \( \alpha \)-closed surjection and \((X, \mathcal{F})\) is normal, then \((Y, \mathcal{U})\) is normal.

Proof: Let \( A \) and \( B \) be closed sets of \((Y, \mathcal{U})\). Then there are open disjoint sets \( U \) and \( V \) in \((X, \mathcal{F})\) such that \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \), by normality of \((X, \mathcal{F})\). By Proposition 6 there are disjoint \( \alpha \)-sets \( C \) and \( D \) in \((Y, \mathcal{U})\) such that \( A \subseteq C \) and \( B \subseteq D \) so that, by Lemma 3, \( \text{int} (\text{cl} (\text{int} C)) \) and \( \text{int} (\text{cl} (\text{int} D)) \) are disjoint open sets in \((X, \mathcal{F})\) containing \( A \) and \( B \) respectively.

References