Indian J. pure appl. Math., 17 (9) : 1101-1105, September 1986

ON FEEBLY CLOSED MAPPINGS

SINA GREENWOOD AND IVAN L. REILLY*

Department of Mathematics and Statistics, University of Auckland
Auckland, New Zealand

(*Received* 22 *August* 1985)

Recently the notion of feebly closed mappings was introduced. In this paper this concept is shown to coincide with the notion of α-closed mappings. Furthermore it is shown that if the codomain is appropriately retopologised the concept of feebly closed mappings coincides with the usual notion of closed mappings. Some properties of α-closed mappings are investigated.

1. INTRODUCTION

Let S be a subset of topological space $(X, \mathcal{T})$. We denote the closure of S and the interior of S with respect to $\mathcal{T}$ by $\mathcal{T}$ cl S and $\mathcal{T}$ int S respectively, although we may suppress the $\mathcal{T}$ when there is no possibility of confusion. We denote the topology induced by $\mathcal{T}$ on S, by $\mathcal{T}_S$. Njåstad[6] introduced the concept of an α-set in $(X, \mathcal{T})$. A subset S of $(X, \mathcal{T})$ is called an α-set if $S \subset \mathcal{T}$ (int $\mathcal{T}$ cl ($\mathcal{T}$ int S)). The notions of semi-open set and preopen set were introduced by Levine[3] and Mashhour *et al.*[5] respectively. A subset S of $(X, \mathcal{T})$ is called a semi-open set (respectively preopen set) if $S \subset \mathcal{T}$ cl ($\mathcal{T}$ int S) (respectively $S \subset \mathcal{T}$ int ($\mathcal{T}$ cl S)). The complements of an α-set, a semi-open set and a preopen set are called α-closed, semi-closed and preclosed respectively. Following Njåstad[6] we denote the family of all α-sets in $(X, \mathcal{T})$ by $\mathcal{T}^{\alpha}$. Njåstad proved that $\mathcal{T}^{\alpha}$ is a topology on X. As any open set in $(X, \mathcal{T})$ is an α-set, $\mathcal{T} \subset \mathcal{T}^{\alpha}$ in the lattice of topologies on the set X. If A is a subset of $(X, \mathcal{T})$, then the intersection of all semiclosed sets containing A is called the semi closure of A, and is denoted s cl A. The largest semi-open set contained in A is denoted by s int A. Maheshwari and Tapi[7] defined A to be a feebly open set in $(X, \mathcal{T})$ if there is an open set U such that $U \subset A \subset s$ cl U. The complement of a feebly open set is called a feebly closed set.

2. FEEBLY CLOSED MAPS

The concepts of α-closed and feebly closed mappings have been introduced by Mashhour *et al.*[5] and Maheshwari and Jain respectively.

*The second author wishes to acknowledge the support of the University of Auckland Research Fund.

Definition 1—A function $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called

(i) feebly closed if the image of each closed set in X is feebly closed in Y;

(ii) α-closed if the image of each closed set in X is α-closed in Y.

Lemma 1—Let A be a subset of $(X, \mathcal{T})$. Then $s\text{int}(\text{cl } A) = \text{cl}(\text{int}(\text{cl } A))$.

PROOF : Notice that cl (int (cl A)) is a semi-open set since cl (int (cl A)) = cl (int (cl (int (cl A)))), and cl (int (cl A)) $\subset$ cl A. Therefore cl (int Cl A)) $\subset$ s int (cl A).

Conversely, if U is any semi-open set contained in cl A, then $U \subset \text{cl}(\text{int } U) \subset \text{cl}(\text{int}(\text{cl } A))$ and therefore s int (cl A) $\subset$ cl (int (cl A)).

Maheshwari and Jain[6] in Lemma 3 of their paper showed that a subset A of $(X, \mathcal{T})$ is feebly closed if and only if s int (cl A) $\subset A$.

Proposition 1—If A is a subset of $(X, \mathcal{T})$, then A is feebly closed if and only if A is α-closed.

PROOF : It follows from the definitions of an α-set and an α-closed set that a subset A of $(X, \mathcal{T})$ is α-closed if and only if cl (int (cl A)) $\subset A$.
Since cl (int (cl A)) $\subset A$ if and only if s int (cl A) $\subset A$, by Lemma 1, A is α-closed if and only if A is feebly closed.

An alternative proof of this result has been given in Proposition 1 of Janković and Reilly[2], who proved that if A is a subset of $(X, \mathcal{T})$, then A is feebly open if and only if A is an α-set.

Proposition 2 follows immediately from Proposition 1 and Definition 1.

Proposition 2—The following are equivalent :

(1) $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is feebly closed;

(2) $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed;

(3) $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^{\alpha})$ is closed.

If the codomain space of a feebly closed mapping f is retopologised in an obvious way, then f is simply a closed mapping. This observation puts the notion of feebly closed mappings into a more natural setting, and enables us to provide immediate proofs of some of the results in Maheshwari and Jain[6]. For example, Propositions 5 and 6 of Maheshwari and Jain[6] are well-known results (Murdeshwar[4], Theorem 4.26, P. 96 and Theorem 4.28, p. 26) restated in this setting.

3. α-CLOSED MAPPINGS

The following classes of generalized closed mappings were introduced in Mashhour *et al.*[5] and Noiri[9].

Definition 2—A function $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is called

(i) semi-closed if the image of each closed set in X is semi-closed in Y;

(ii) preclosed if the image of each closed set in X is preclosed in Y.

It is shown in Theorem 3 of Reilly and Vamanamurthy[10] that a subset of $(X, \mathcal{T})$ is an α-set if and only if it is semi-open and preopen. Thus we have the following result.

Proposition 3—A function $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed if and only if it is semi-closed and preclosed.

Examples 1 and 2 show that the separate converses are not in general true.

Example 1—Let $X = \{a, b, c\}$ and define the topologies $\mathcal{T}$ to be the discrete topology and $\mathcal{U} = \{\phi, x, \{a\}, \{c\}, \{a, c\}\}$. We define $f: (X, \mathcal{T}) \to (X, \mathcal{U})$ by $f(a) = f(b) = f(c) = a$. Then f is preclosed but not α-closed since $\{a\}$ is preclosed in $(X, \mathcal{U})$ but not α-closed in $(X, \mathcal{U})$.

Example 2—If $\mathcal{T}$ is the discrete topology and $\mathcal{U}$ is the indiscrete topology then $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is semi-closed but not α-closed.

Andrijević[1] showed that if M is a subset of $(X, \mathcal{T})$ then $(\mathcal{T}_M)^\alpha \subset (\mathcal{T}^\alpha)_M$ (his Theorem 3.2) and if M is preopen then $(\mathcal{T}_M)^\alpha = (\mathcal{T}^\alpha)_M$ (his Theorem 3.6).

Proposition 4—Let $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ and $f(X) \subset Y_1 \subset Y$.

(1) If $f: (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed, then so is $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$.

(2) If Y_1 is preopen and $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed, then $f: (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed.

PROOF: (1) If $f: (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed then $f: (X, \mathcal{T}) \to (Y_1, (\mathcal{U}_{Y_1})^\alpha)$ is closed. By Andrijević's result $f: (X, \mathcal{T}) \to (Y_1, (\mathcal{U}^\alpha)_{Y_1})$ is closed and by Theorem 4.24 (2) of Murdeshwar[4] $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^\alpha)$ is closed. Hence $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed.

(2) If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed then $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^\alpha)$ is closed and by Theorem 4.24, of Murdeshwar,[4] $f: (X, \mathcal{T}) \to (Y_1, (\mathcal{U}^\alpha)_{Y_1})$ is closed. If Y_1 is preopen then by Andrijević's result $f: (X, \mathcal{T}) \to (Y_1, (\mathcal{U}_{Y_1})^\alpha)$ is closed, and so $f: (X, \mathcal{T}) \to (Y_1, \mathcal{U}_{Y_1})$ is α-closed.

Proposition 5—If f is an α-closed mapping $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$, $\mathcal{T}_1 \subset \mathcal{T}$ and $\mathcal{U}^\alpha \subset \mathcal{U}_1^\alpha$ then $f: (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ is α-closed.

PROOF: If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed then $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^{\alpha})$ is closed, and by Theorem 4.23 of Murdeshwar[4], $f: (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1^{\alpha})$ is closed. Hence $f: (X, \mathcal{T}_1) \to (Y, \mathcal{U}_1)$ is α-closed.

Proposition 6—If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is an α-closed mapping and $B, C \subset Y$, then if $f^{-1}(B)$ and $f^{-1}(C)$ have disjoint neighbourhoods, B and C have disjoint neighbourhoods in $(Y, \mathcal{U}_{\alpha})$.

PROOF: If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed then $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^{\alpha})$ is closed and the result follows from Murdeshwar[4] Theorem 4.28 (2).

Proposition 7—Let $f: X \to (Y, \mathcal{U})$ and let X be given the preimage topology $\mathcal{T}$. Then f is α-closed if and only if $f(X)$ is α-closed in $(Y, \mathcal{U})$.

PROOF: One implication, namely if f is α-closed then $f(X)$ is α-closed in $(Y, \mathcal{U})$ is clear since X is closed in the preimage topology.

Conversely, if $f(X)$ is α-closed in $(Y, \mathcal{U})$ then $f(X)$ is closed in $(Y, \mathcal{U}^{\alpha})$. Let $\mathcal{T}_{\alpha}$ be a preimage topology on X for $f: X \to (Y, \mathcal{U}^{\alpha})$.

Then, by Theorem 4.30 of Murdeshwar[4] $f: (X, \mathcal{T}_{\alpha}) \to (Y, \mathcal{U}^{\alpha})$ is closed. Since $\mathcal{U} \subset \mathcal{U}^{\alpha}$ and therefore $\mathcal{T} \subset \mathcal{T}_{\alpha}$, $f: (X, \mathcal{T}) \to (Y, \mathcal{U}^{\alpha})$ is closed, and so $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is α-closed.

It is well known that the T_1 property is preserved under closed mappings. The following example shows that the T_1 property is not preserved under α-closed mappings.

Example 3—Let X be an infinite set and p be a fixed point of X. We define a topology $\mathcal{T}$ on X as follows: for $G \subset X$, $G \in \mathcal{T}$ if $G = \phi$ or $G = X$ or $X - G$ is finite. We define a topology $\mathcal{U}$ on X as follows: for each $G \subset X$, $G \in \mathcal{U}$ if (i) $G = \phi$ or $G = X$, or (ii) $G \subset X - \{p\}$ and $X - G$ is finite. $(X, \mathcal{T})$ is T_1 but $(X, \mathcal{U})$ is not T_1 since for any point x distinct from p the only open set containing p, namely X, contains x. Let $f: (X, \mathcal{T}) \to (X, \mathcal{U})$ be the identity function. Then f is α-closed since if A is a closed subset of $(X, \mathcal{T})$ then either A is a closed subset of $(X, \mathcal{U})$ and therefore α-closed in $(X, \mathcal{U})$, or A is finite, nonempty and $p \notin A$. In this case $\mathcal{U}$ cl ($\mathcal{U}$ int ($\mathcal{U}$ cl A)) $= \phi \subset A$ and therefore A is an α-closed subset of $(X, \mathcal{U})$.

The following proposition is a generalization of the well known result that regularity is preserved under continuous, open and closed surjections (Murdeshwar[4] Theorem 12.14, p. 206).

Lemma 2—If U and V are subsets of $(X, \mathcal{T})$, U is open and $U \subset V$, then cl $U \subset$ cl (int (cl V)).

PROOF: If U is open and $U \subset V$, then $U \subset$ cl V so that $U \subset$ int (cl V). Therefore cl $U \subset$ cl (int (cl (V)).

Proposition 8—If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is an open, continuous, α-closed surjection and $(X, \mathcal{T})$ is regular, then $(Y, \mathcal{U})$ is regular.

PROOF : Let $p \in Y$ and U be an open set in $(Y, \mathcal{U})$ containing p. Let $x \in X$ such that $f(x) = p$. Since $(X, \mathcal{T})$ is regular there is an open set V in $(X, \mathcal{T})$ such that $x \in V \subset \mathcal{T}\,\mathrm{cl}\, V \subset f^{-1}(U)$ so that $p \in f(V) \subset f(\mathcal{T}\,\mathrm{cl}\, V) \subset U$. Since f is α-closed, $f(\mathcal{T}\,\mathrm{cl}\, V)$ is α-closed and since f is open $f(V)$ is open so that, by Lemma 2, $\mathcal{U}\,\mathrm{cl}\, f(V) \subset \mathcal{U}\,\mathrm{cl}\,(\mathcal{U}\,\mathrm{int}\,(\mathcal{U}\,\mathrm{cl}\, f(\mathcal{T}\mathrm{cl}\ f(V))) \subset U$ and therefore $p \in f(V) \subset \mathcal{U}\,\mathrm{cl}\, f(V) \subset U$.

It is well known that normality is preserved under closed, continuous surjections Murdeshwar[4] Theorem 15.3 (1). The following proposition is a generalization of this result.

Lemma 3—If U and V are subsets of $(X, \mathcal{T})$ and $U \cap V = \phi$ then int (cl (int U)) $\cap$ int (cl (int V) $= \phi$.

PROOF : If $U \cap V = \phi$ then int $U \cap$ int $V = \phi$, so that int $U \cap$ cl (int V) $= \phi$. Therefore int $U \cap$ int (cl (int V)) $= \phi$ which implies that cl (int U) $\cap$ int (cl (int V)) ϕ, so that we have int (cl (int U)) $\cap$ int (cl int V)) $= \phi$.

Proposition 9—If $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a continuous, α-closed surjection and $(X, \mathcal{T})$ is normal, then $(Y, \mathcal{U})$ is normal.

PROOF : Let A and B be closed sets of $(Y, \mathcal{U})$. Then there are open disjoint sets U and V in $(X, \mathcal{T})$ such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$, by normality of $(X, \mathcal{T})$. By Proposition 6 there are disjoint α-sets C and D in $(Y, \mathcal{U})$ such that $A \subset C$ and $B \subset D$ so that, by Lemma 3, int (cl (int C)) and int (cl (int D)) are disjoint open sets in $(Y, \mathcal{U})$ containing A and B respectively.

REFERENCES

1. D. Andrijevic, *Mat. Vesnik* **36** (1984), 1–10.
2. D. S. Jankovic, and I. L. Reilly, *Indian J. pure appl. Math.* **16** (1985).
3. N. Levine, *Amer. Math. Mthly.* **70** (1963), 36–41.
4. M. G. Murdeshwar, *General Topology*. Halstead Press, New York, 1983.
5. A. S. Mashhour, I. A. Hasanein, and S. N. El-Deeb, *Acta Math.Hung.* **41** (1983), 213–18.
6. S. N. Maheshwari, and P. C. Jain, *J. Sci. Res.* **4** (1982), 67–70.
7. S. N. Maheshwari, and U. D. Tapi, *Madhya Bharati J. Univ. Saugar*, (1978-79) (to appear).
8. O. Njåstad, *Pac. J. Math.* **15** (1965), 961–70.
9. T. Noiri, *Atti. Acad. Nat. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **54** (1973), 412–15.
10. I. L. Reilly and, M. K. Vamanamnrthy, *Acta Math. Hung.* **45** (1985), 27–32.