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ON α-STARLIKE AND α-CLOSE-TO-CONVEX FUNCTIONS WITH RESPECT TO n-SYMMETRIC POINTS

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We define the class $K_n(\alpha; h)$—the class of α-starlike functions with respect to n symmetric points—consisting of $f(z) = z + a_2z^2 + \cdots$ satisfying $\frac{f(z)\,f'(z)}{z} \neq 0$ in E and $\frac{\alpha z(zf'(z))' + (1-\alpha)\,z f'(z)}{\alpha z f_n'(z) + (1-\alpha) f_n(z)} \prec h(z)$, $\alpha \geqslant 0$, where $h(z)$ is a given convex univalent function in E with $h(0) = 1$ $\mathrm{Re}\, h(z) > 0$ in E and $f_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{-j} f(\omega^j z)$ with $\omega = \exp(2\pi i/n)$. In this paper we prove theorems which establish that $K_n(\alpha; h) \subset K_n(0; h)$, $K_n(\alpha; h)$ is closed under an integral operator and the coefficient estimate for this class. Also we define two class $C_n(\alpha; h)$ of functions and investigate the properties of this class.

INTRODUCTION

Let $E = \{z : |z| < 1\}$ be the open unit disc in $\mathbb{C}$, $H(E)$ denote the class of functions $f(z)$ holomorphic in E. Let $A = \{f \in H(E) : f(0) = f'(0) - 1 = 0\}$. For a given positive integer n let $\omega = \exp(2\pi i/n)$ and

$$f_n(z) = \frac{1}{n}\sum_{j=0}^{n-1} \omega^{-j} f(\omega^j z),\ z \in E. \qquad \ldots(1)$$

Here we introduce new subclasses of S-the class of normalized univalent functions and study certain properties of these classes. First let us define the class $K_n(\alpha; h)$ which unifies the classes of convex univalent functions with respect to symmetric points and starlike univalent functions with respect to symmetric points.

Definition 1—The function $f(z) \in A$ and $\frac{f(z).f'(z)}{z} \neq 0$ in E is said to be α-starlike with respect to n-symmetric points if it satisfies

$$\frac{\alpha z\,(zf'\,z))' + (1-\alpha)\,zf'\,(z)}{\alpha z\,f_n'\,(z) + (1-\alpha) f_n(z)} \prec h\,(z), \quad \text{for } \alpha \geqslant 0$$

where $h\,(z)$ is a given convex univalent function in E with $h\,(0) = 1$, $\operatorname{Re} h\,(z) > 0$ and $\prec$ means subordination. Let us denote the class of such functions as $K_n\,(\alpha;h)$.

When $n = 1$ and $h\,(z) = \dfrac{1-z}{1+z}$ this reduces to the class $K\,(\alpha)$ of α-starlike functions studied by Pascu and Podaru[7]. Also $K_2\left(1, \dfrac{1-z}{1+z}\right)$ coincides with the class Cs-defined by Das and Singh[1], $K_2\left(0, \dfrac{1-z}{1+z}\right)$ is the class of starlike functions with respect to symmetric points investigated by Sakaguchi[8]. Of course $K_n\,(0; h)$ is the class of starlike functions with respect to n-symmetric points of Mocanu[3]. Also here we define another class $C_n\,(\alpha; h)$ of α-close-to-convex functions with respect to n-symmetric points.

Definition 2—A function $f\,(z) \in A$ such that $\dfrac{f\,(z)\,f'\,(z)}{z} \neq 0$ in E is said to be α-close-to-convex with respect to n-symmetric points if it satisfies

$$\frac{\alpha z\,(zf'\,(z))' + (1-\alpha)\,zf'\,(z)}{\alpha z\phi_n'\,(z) + (1-\alpha)\,\phi_n\,(z)} \prec h\,(z), \text{ for } \alpha \geqslant 0$$

where $\phi_n\,(z) = \dfrac{1}{n}\sum_{j=0}^{n-1} \omega^{-j}\,\phi\,(\omega^j z)$ with $\phi\,(z) \in K_n\,(\alpha;\ h)$ and $h\,(z)$ is as defined in Definition 1.

This class generalizes the classes defined by Pascu[6] and Das and Singh[1]. Let us investigate these two classes in this paper. In establishing our main results regarding these classes we often make use of the following lemma due to Padmanabhan and Parvatham[4].

Lemma A—Let $\beta, \gamma \in \mathbb{C}$, $h \in H\,(E)$ be convex univalent in E with $h\,(0) = 1$ and $\operatorname{Re}\,(\beta h\,(z) + \gamma) > 0$, $z \in E$ and let $q \in H\,(E)$ with $q\,(0) = 1$ and $q\,(z) \prec h\,(z)$, $z \in E$. If $p\,(z) = 1 + p_1 z + \ldots$ in analytic in E, then

$$p\,(z) + \frac{zp'\,(z)}{\beta\,q\,(z) + \gamma} \prec h\,(z) \Rightarrow p\,(z) \prec h\,(z).$$

Now, we prove a lemma which we use in the sequel.

Lemma—Let $f\,(z) \in K_n\,(\alpha;\ h)$. Then $f_n\,(z)$ defined by (1) is in $K_1\,(\alpha;\ h)$. Further

$$\frac{zf_n'\,(z)}{f_n\,(z)} \prec h\,(z) \text{ in } E.$$

PROOF : First let us consider

$$\frac{\alpha z\,(z f_n'(z))' + (1-\alpha)\, z f_n'(z)}{\alpha z f_n'(z) + (1-\alpha) f_n(z)} = \frac{\frac{1}{n}\sum_{j=0}^{n-1} \alpha z\,(zf'(\omega^j z))' + (1-\alpha) zf'(\omega^j z)}{\alpha z f_n(z) + (1-\alpha) f_n(z)}$$

$$= \frac{1}{n}\sum_{j=0}^{n-1} \frac{\alpha z\,(zf'(\omega^j z))' + (1-\alpha\, zf'(\omega^j z)}{\alpha z f_n'(z) + (1-\alpha\, f_n(z)}.$$

Since $f(z) \in K_n(\alpha; h)$ each term under the summation on the right side of the above equality is subordinate to $h(z)$. Hence there exist ζ_j's in E such that

$$\frac{\alpha z\left(zf_n'(z)\right)' + (1-\alpha)\, z f_n'(z)}{\alpha z f_n'(z) + (1-\alpha) f_n(z)} = \frac{1}{n}\sum_{j=0}^{n-1} h(\zeta_j) = h(\zeta_0)$$

$\zeta_0 \in E$ since $h(E)$ is convex. Thus $f_n(z) \in K_1(\alpha, h)$.

Since,

$$\frac{\alpha z\left(zf_n'(z)\right)' + (1-\alpha)\, zf_n'(z)}{\alpha z f_n'(z) + (1-\alpha) f_n(z)} = p(z) + \frac{\alpha z p'(z)}{(1-\alpha) + \alpha p(z)}$$

where $p(z) = \dfrac{zf_n'(z)}{f_n(z)}$, an application of a result by Eenigenburg *et al.*[2] now yields

$$\frac{zf_n'(z)}{f_n(z)} = p(z) \prec h(z).$$

Theorem 1—We have the following inclusion relation :

$$K_n(\alpha; h) \subset K_n(0; h).$$

PROOF : Let $f(z) \in K_n(\alpha; h)$, $p(z) = \dfrac{zf'(z)}{f_n(z)}$ and $q(z) = \dfrac{zf_n'(z)}{f_n(z)}$.

Then

$$\frac{\alpha z\,(zf'(z))' + (1-\alpha)\, zf'(z)}{\alpha z f_n'(z) + (1-\alpha) f_n(z)}$$

(*equation continued on p.* 1117)

$$= \frac{\alpha z\,[f_n'(z)\,p(z) + p'(z)\,f_n(z)] + (1-\alpha)\,[p(z)\,f_n(z)]}{\alpha z f_n'(z) + (1-\alpha) f_n(z)}$$

$$= \frac{\alpha z p'(z) + \dfrac{\alpha z f_n'(z)}{f_n(z)}\,p(z) + (1-\alpha)\,p(z)}{\alpha z \dfrac{f_n'(z)}{f_n(z)} + (1-\alpha)}$$

$$= \frac{\alpha z p'(z) + p(z)\left[(1-\alpha) + \dfrac{\alpha z f_n'(z)}{f_n(z)}\right]}{(1-\alpha) + \dfrac{\alpha z f_n'(z)}{f_n(z)}}$$

$$= \frac{\alpha z p'(z)}{(1-\alpha) + \alpha q(z)} + p(z) \prec h(z)$$

since $f(z) \in K_n(\alpha; h)$. Here $q(z) \prec h(z)$ by the above lemma. Now an application of lemma A gives $p(z) \prec h(z)$ in E which completes the proof of the theorem.

Remark 1 : When $n = 1$ and $h(z) = \dfrac{1-z}{1+z}$ we get a result of Pascu[5] as a particular case of our result.

Now, we establish a theorem which shows that this class is closed under an integral operator.

Theorem 2—$F(z) = \dfrac{1}{\alpha z^{1/\alpha - 1}} \displaystyle\int_0^z t^{(1/\alpha)-2} f(t)\,dt \in K_n(\alpha; h)$

whenever $f(z) \in K_n(\alpha; h)$ for $\alpha > 0$.

PROOF : Consider $F_n(z)$ defined by (1) with $F(z)$ in the place of $f(z)$. That is, $F_n(z) = \dfrac{1}{n} \displaystyle\sum_{j=0}^{n-1} \omega^{-j} F(\omega^j z)$. It is easy to see that $F_n(z) = \dfrac{1}{\alpha z^{(1/\alpha)-1}} \displaystyle\int_0^z t^{(1/\alpha)-2} f_n(t)dt$.

Differentiating this with respect to z and simplifying we get

$$\alpha z F_n'(z) + (1-\alpha) F_n(z) = f_n(z). \qquad \ldots(2)$$

From the definition of $F(z)$ we get, $\alpha z F'(z) + (1-\alpha) F(z) = f(z)$ which on differentiation gives

$$\alpha z (zF'(z))' + (1-\alpha) zF'(z) = zf'(z). \qquad \ldots(3)$$

From (2) and (3) we get

$$\frac{\alpha z (zF'(z))' + (1-\alpha) zF'(z)}{\alpha z F_n'(z) + (1-\alpha) F_n(z)} = \frac{zf'(z)}{f_n(z)} \prec h(z)$$

because of Theorem 1 and the fact that $f(z) \in K_n(\alpha;h)$. This implies that $F(z) \in K_n(\alpha;h)$.

We now obtain the coefficient estimates for the class $K_n(\alpha;h)$.

Theorem 3—Let $f(z) = z + \sum_{i=2}^{\infty} a_i z^i \in K_n(\alpha;h)$. Then $\forall\ k \geqslant 1,\ n = 1, 2, \ldots$

$$|a_{kn+1}| \leqslant \frac{|h_1| \left(1+\frac{|h_1|}{n}\right)\left(1+\frac{|h_1|}{2n}\right)\ldots\left(1+\frac{|h_1|}{(k-1)n}\right)}{kn(kn\alpha+1)} \qquad \ldots(4)$$

for i satisfying $(k-1)n + 1 < i < kn + 1$, $\forall\ k \geqslant 1, n = 1, 2, \ldots$

$$|a_i| \leqslant \frac{|h_1| \left(1+\frac{|h_1|}{n}\right)\left(1+\frac{|h_1|}{2n}\right)\ldots\left(1+\frac{|h_1|}{(k-1)n}\right)}{i(i\alpha+1-\alpha)}$$

where $h(z) = 1 + h_1 z + \ldots.$

PROOF: Let $\dfrac{\alpha z (zf'(z))' + (1-\alpha) zf'(z)}{\alpha z f_n'(z) + (1-\alpha) f_n(z)} = p(z) = 1 + p_1 z + p_2 z^2 + \ldots$

where $f_n(z) = \frac{1}{n} \sum_{j=0}^{n-1} \omega^{-j} f(\omega^j z)$ and so $f_n'(z) = \frac{1}{n} \sum_{j=0}^{n-1} f'(\omega^j z)$.

Since $f(z) \in K_n(\alpha;h)$, $p(z) \prec h(z) = 1 + h_1 z + h_2 z^2 + \ldots.$ It is well known that $\forall\ i \geqslant 2$, $|p_i| \leqslant |h_1|$. Now,

$$\alpha z (zf'(z))' + (1-\alpha) zf'(z) = (1 + p_1 z + p_2 z^2 + \ldots)(\alpha z f_n'(z) + (1-\alpha) f_n(z))$$

(equation continued on p. 1119)

$$= (1 + p_1 z + \ldots)\left(z + \sum_{i=2}^{\infty} a_i \left(\frac{\alpha(i-1)+1}{n}\right)\right)\left(\sum_{j=0}^{n-1} \omega^{(i-1)j} z^j\right).$$

Comparing the coefficients on either side,

$$\begin{aligned} |a_i|(i^2\alpha + i(1-\alpha)) \leqslant |h_1| + |h_1||a_2|\left(\frac{\alpha+1}{n}\right)\sum_{j=0}^{n-1}\omega^j \\ + |h_1||a_3|\left(\frac{2\alpha+1}{n}\right)\sum_{j=0}^{n-1}\omega^{2j} \\ + \ldots + |h_1||a_{i-1}|\left(\frac{(i-2)\alpha+1}{n}\right) \\ \times \sum_{j=0}^{n-1}\omega^{(i-2)j} \\ + |a_i|\left(\frac{(i-1)a+1}{n}\right)\sum_{j=0}^{n-1}\omega^{(i-1)j}. \end{aligned} \qquad \ldots(5)$$

Also we make use of the fact that $\sum_{j=0}^{n-1} \omega^{(i-1)j} = n$ when $i = kn + 1$, $\forall\ k > 1$; $\sum_{j=0}^{n-1} \omega^{(i-1)j} = 0$ when $i \neq kn + 1$, $\forall\ k \geqslant 1$.

Hence let us consider the two cases separately.

Case (*i*) —First let us consider the case $i = kn + 1$. Then we establish (4) for all $k > 1$, by the method of induction, In this case (5) reduces to

$$\begin{aligned} kn(kn\alpha + 1)|a_{kn+1}| \leqslant |h_1| + |h_1|(n\alpha + 1)|a_{n+1}| \\ + |h_1|(2n\alpha + 1)|a_{2n+1}| + \ldots + \\ + |h_1|((k-1)k\alpha + 1)|a_{(k-1)n+1}|. \end{aligned}$$

Let us assume that the estimate (4) is true for $k = 1, 2, \ldots, (j-1)$. Hence the above inequality yields,

$$\begin{aligned} |jn(jn\alpha + 1)|a_{jn+1}| \leqslant |h_1| + |h_1|\frac{|h_1|}{n} + \frac{|h_1||h_1|\left(1 + \frac{|h_1|}{n}\right)}{2n} \\ + \ldots + \frac{|h_1||h_1|\left(1 + \frac{|h_1|}{n}\right)\ldots\left(1 + \frac{|h_1|}{(j-2)n}\right)}{(j-1)n} \end{aligned}$$

which on simplification gives

$$|a_{jn+1}| \leqslant \frac{|h_1|\left(1+\frac{|h_1|}{n}\right)\left(1+\frac{|h_1|}{2n}\right)+\ldots+\left(1+\frac{|h_1|}{(j-1)n}\right)}{jn\,(jn\alpha+1)}.$$

This shows that (4) is true for $k = j$ also. It is obvious from (5) that (4) is true for $k = 1$. Hence it is true for $\forall\ k > 1$.

Case (ii)—Let $i \neq jn + 1$ for $j \geqslant 1$. Then there exists a k such that $(k-1)n+1 < i < kn+1$. Here (5) reduces to

$$\begin{aligned} i\,(i\alpha+1-\alpha)\,|a_i| \leqslant\ & |h_i| + |h_1|\,(n\alpha+1)\,|a_{n+1}| \\ & + |h_1|\,(2n\alpha+1)\,|a_{2n+1}| \\ & + \ldots + |h_1|\,((j-1)n\alpha+1)\,|a_{(j-1)n+1}|. \end{aligned}$$

Using (4) the above inequality becomes for $(k-1)n+1 < i < kn+1$

$$\begin{aligned} i\,(i\alpha+1-\alpha)\,|a_i| \leqslant\ & |h_1| + |h_1|\frac{|h_i|}{n} + \frac{|h_1||h_1|\left(1+\frac{|h_1|}{n}\right)}{2n} \\ & + \ldots + |h_1||h_1|\left(1+\frac{|h_1|}{n}\right)\ldots\left(1+\frac{|h_1|}{(k-2)n}\right) \end{aligned}$$

which on simplification gives

$$|a_i| \leqslant \frac{|h_1|\left(1+\frac{|h_1|}{n}\right)\left(1+\frac{|h_1|}{2n}\right)\ldots\left(1+\frac{|h_1|}{(k-1)n}\right)}{i\,(i\alpha+1-\alpha)}.$$

Remark 2 : When h is a general convex univalent function we can not ascertain anything about the sharpness of the coefficient inequality. However when $n = 2$, $\alpha = 0$ and $h(E)$ is the half plane Re $z > 0$, we get $|a_i| \leqslant 1$, $\forall\ i \geqslant 2$, the sharp estimate due to Sakaguchi[8]. When $n = 2$, $\alpha = 1$ and $h(E)$ is the half plane Re $z > 0$ we have for $\forall\ k \geqslant 1$, the sharp estimate $|a_k| \leqslant \frac{1}{k}$, a result of Das and Singh[1].

Theorem 4—We have $C_n(\alpha;h) \subset C_n(0;h)$.

PROOF : Let $f(z) \in C_n(\alpha;h)$. Setting $p(z) = \frac{zf(z)}{\phi_n(z)}$ and $q(z) = z\frac{\phi_n'(z)}{\phi_n(z)}$ we have

$$\frac{\alpha z\,(zf'(z))' + (1-\alpha)\,zf'(z)}{\alpha z\phi_n'(z) + (1-\alpha)\,\phi_n(z)} = \frac{\alpha z p'(z) + p(z)\left(\alpha z\frac{\phi_n'(z)}{\phi_n(z)} + 1 - \alpha\right)}{\alpha z\frac{\phi_n'(z)}{\phi_n(z)} + (1-\alpha)}$$

(*equation continued on p.* 1121)

$$= p(z) + \frac{\alpha z p'(z)}{\alpha q(z) + (1-\alpha)} \prec h(z)$$

since $f(z) \in C_n(\alpha;h)$. Here $q(z) \prec h(z)$ (by our lemma). Again an application of lemma A yields $p(z) = \frac{zf'(z)}{\phi_n(z)} \prec h(z)$ which establishes the theorem.

Now let us get an integral representation for functions belonging to the class $C_n(\alpha;h)$.

Theorem 5 – A function $f(z) \in C_n(\alpha;h)$ if and only if there exists a function $G(z)$ in $H(E)$ with $G(o)=0$ such that $\frac{zG'(z)}{G(z)} \prec h(z)$ and an analytic function $p(z)$ with $p(0) = 1$ and $p(z) \prec h(z)$ in E such that

$$f'(z) = \frac{1}{\alpha z^{1/\alpha}} \int_0^z p(t)\, G(t)\, t^{(1/\alpha)-2}\, dt, \text{ if } \alpha \neq 0$$

and

$$f'(z) = \frac{p(z)\, G(z)}{z}, \text{ if } \alpha = 0.$$

PROOF : Let $f(z) \in C_n(\alpha;h)$. Then there exists a

$$\phi \in K_n(\alpha;h) \text{ such that } \frac{\alpha z (zf'(z))' + (1-\alpha) zf'(z)}{\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z)} \prec h(z)$$

where $\phi_n(z)$ is defined by (1). By our lemma $\phi_n \in K_1(\alpha;h)$; That is,

$$\frac{\alpha z \left(z\, \phi_n'(z)\right)' + (1-\alpha)\, z\phi_n'(z)}{\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z)} \prec h(z).$$

Hence we can write

$$\frac{\alpha z \left(z\, \phi_n'(z)\right)' + (1-\alpha)\, z\, \phi_n'(z)}{\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z)} = \frac{zG'(z)}{G(z)}$$

where $\frac{zG'(z)}{G(z)} \prec h(z)$ which on integration gives,

$$\alpha z\, \phi_n'(z) + (1-\alpha)\, \phi_n(z) = G(z)$$

and so,

$$\frac{\alpha z\,(zf'(z))' + (1-\alpha)\,zf'(z)}{G(z)} = p(z) \prec h(z)$$

that is,

$$\alpha z\,(zf'(z))' + (1-\alpha)\,zf'(z) = p(z)\,G(z).$$

When $\alpha \neq 0$ multiplying by $\frac{z^{(1/\alpha)-2}}{\alpha}$ and integrating we get

$$f'(z) = \frac{1}{\alpha z^{1/\alpha}} \int_0^z p(t)\,G(t)\,t^{(1/\alpha)-2}\,dt.$$

Conversely it is easily seen that if $f(z)$ has the above integral representation then $f(z) \in C_n(\alpha;h)$

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