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INVERSE AND REVERSIBILITY OF QUASIMATRIX TRANSFORMATION

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The object of the paper is to determine conditions under which the quasi method associated with an infinite matrix admits an inverse transformation.

1. INTRODUCTION AND DEFINITION

Vermes[8] pointed out that there is a close relationship between the summability properties of a matrix $A = (a_{nk})$ regarded as a sequence-to-sequence transformation and those of its transpose $A^* = (a_{kn})$ regarded as a series-to-series transformation. We call A^* the quasi method associated with A.

Given an infinite series† Σa_n and the matrix $A = (a_{nk})$, the quasi method A^* is to be 'applicable' if the A^*-transformation

$$b_n = \sum_{k=0}^{\infty} a_{kn}\, a_k \qquad \ldots(1)$$

exists for each $n \geqslant 0$. The series Σa_n is said to be A^* summable (or summable by the quasi-method associated with A) to the value s, if further

$$\Sigma b_n = s. \qquad \ldots(2)$$

When A is a normal matrix, then A^* is in general a row infinite matrix and therefore it is relatively more difficult to work with.

The generalised Nörlund matrix (N, p, q) defined by

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}\, q_k}{r_n} & (k \geqslant n) \\ 0 & (k > n) \end{cases}$$

where $r_n = p_0\, q_n + \ldots + p_n\, q_0 \neq 0$ $(n \geqslant 0)$, is normal provided that $q_n \neq 0$. In the case $q_n = 1$ (all n) then the method reduces to the familiar Nörlund method (N, p), see Hardy[2] (p. 64).

Kuttner[3] defined quasi-Cesàro as quasi-Hausdorff transformation. Thorpe[7] has considered the transpose of the Nörlund matrix regarded as giving a series-to-series transformation in the case $p_n \in \mathcal{M}$, i.e.

† Summation without limits means from 0 to ∞.

$$p_n > 0, \frac{p_{n+1}}{p_n} \leqslant \frac{p_{n+2}}{p_{n+1}} \leqslant 1 \, (n = 0, 1, 2, \ldots).$$

Mohapatro[4,5] has considered the transpose of the generalised Nörlund matrix $(N, p. q)$ in the case $p_n \in \mathcal{M}$, $q_n > 0$, $q_{n+1} \geqslant q_n$.

2. Statlment of Theorems

Generalising the results of Thorpe[7] and Mohapatro[6] we obtain the inverse of quasi-matrix transformation in Theorem 1 below whereas in Theorem 2 we determine conditions under which it is possible to recapture the original transformation from its inverse transformation.

Before we state our theorems, we require the following notation :

We write $A \in \mathcal{P}$ if A is normal and satisfies

$$a_{nk} > 0 \, (k \leqslant n); \frac{a_{n+1,k+1}}{a_{n,k+1}} \leqslant \frac{a_{n+1,k}}{a_{nk}} \, (0 \leqslant k \leqslant n-1). \qquad \ldots(3)$$

We write $A \in \mathcal{P}^*$ if $A \in \mathcal{P}$ and

$$a_{nk} \leqslant a_{n,k+1} \, (0 \leqslant k \leqslant n-1). \qquad \ldots(4)$$

Theorem 1—Let $A \in \mathcal{P}$. Then the A^* transformation (when applicable) has an inverse whose matrix is given by the transpose of the inverse of the A transformation; that is, if b_n is given by (1) then

$$a_n = \sum_{k=n}^{\infty} a_{kn}^{-1} b_k. \qquad \ldots(5)$$

A matrix B, regarded as a series-to-series transformation, is called 'reversible,' if for any $b = (b_n)$ with Σb_n convergent, there is a 'unique' $a = (a_n)$ such that $Ba = b$. In the following theorem, we consider the reversibility of A^* transformation, regarded as a series-to series transformation.

Theorem 2—Let $A \in \mathcal{P}^*$. Then A^* transformation is reversible; more generally, if for any $b_n = o(1)$, then there exist a unique $a = (a_n)$ such that (1) holds.

For the proof of the theorem, we require the following lemmas :

Lemma 1 (a)—Let $A \in \mathcal{P}$. Then (see Peyerimhoff[6], p. 33)

(i) $a_{nn}^{-1} > 0, \forall \, n \geqslant 0; a_{kn}^{-1} \leqslant 0 \, (k > n)$;

(b) Let $A \in \mathcal{P}^*$. Then (see Das[1])

(ii) $\sum_{k=n}^{\infty} |a_{kn}^{-1}| < \infty$ for each $n \geqslant 0$,

(iii) $\sum_{k=n}^{\infty} a_{kn}^{-1} \geqslant 0$ for each $n \geqslant 0$,

(iv) $\sum_{k=m+1}^{\infty} |a_{kn}^{-1}| \leqslant \sum_{k=n}^{m} a_{kn}^{-1}$ $(m \geqslant n)$.

Lemma 2—Let $A \in \mathscr{F}$. Then for $0 \leqslant n < N < M$, we have

$$0 \leqslant \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk} \leqslant \frac{a_{Mn}}{a_{nn}} \leqslant \frac{a_{Mo}}{a_{no}}.$$

PROOF ; Since by Lemma 1 (1), $a_{nn}^{-1} > 0$ and $a_{kn}^{-1} \leqslant 0$ for $k > n$, it follows that

$$a_{nn}^{-1} a_{Mn} \geqslant \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk} \geqslant \sum_{k=n}^{M} a_{kn}^{-1} a_{Mk} = 0$$

by use of the identity (6) as $M > n$. But by (3)

$$\frac{a_{nk}}{a_{nk-1}} \leqslant \frac{a_{mk}}{a_{mk-1}} \quad (1 \leqslant k \leqslant m \leqslant n).$$

Hence it follows that

$$\begin{aligned} a_{nn}^{-1} a_{Mn} = \frac{a_{Mn}}{a_{nn}} &= \frac{1}{a_{nn}} \cdot \frac{a_{Mn}}{a_{M-1}} \cdot \frac{a_{Mn-1}}{a_{Mn-2}} \cdots \frac{a_{M1}}{a_{Mo}} \cdot a_{Mo} \\ &\leqslant \frac{1}{a_{nn}} \cdot \frac{a_{nn}}{a_{nn-1}} \cdot \frac{a_{nn-1}}{a_{nn-2}} \cdots \frac{a_{n1}}{a_{n0}} \cdot a_{M0} \\ &= \frac{a_{M0}}{a_{n0}}. \end{aligned}$$

This completes the proof of the lemma.

3. PROOF OF THEOREM 1

Substituting the value of b_n as given in eqn. (1) and using the identity

$$\sum_{k=n}^{r} a_{kn}^{-1} a_{rk} = \begin{cases} 1 \ (r = n) \\ 0 \ (r > n) \end{cases} \qquad \ldots(6)$$

we have, for any integer $N > n$,

$$\sum_{k=n}^{N} a_{kn}^{-1} b_k = \sum_{k=n}^{N} a_{kn}^{-1} \sum_{r=k}^{\infty} a_{rk}\, a_r$$

$$= \sum_{k=n}^{N} a_{kn}^{-1} \left(\sum_{r=k}^{N} + \sum_{r=N+1}^{\infty} \right) a_{rk}\, a_r$$

$$= \sum_{r=n}^{N} a_r \sum_{k=n}^{r} a_{kn}^{-1} a_{rk} + \sum_{k=n}^{N} a_{kn}^{-1} \sum_{r=N+1}^{\infty} a_{rk}\, a_r$$

$$= a_n + \sum_{r=N+1}^{\infty} a_r \sum_{k=n}^{N} a_{kn}^{-1} a_{rk}. \qquad \ldots(7)$$

So it is obvious from (7) that (5) holds if and only if, for fixed n,

$$\varphi_N = \sum_{r=N+1}^{\infty} a_r \sum_{k=n}^{N} a_{kn}^{-1} a_{rk} \to 0 \text{ as } N \to \infty. \qquad \ldots(8)$$

Since $b_0 = \sum_{k=0}^{\infty} a_{k0}\, a_k$ converges by hypothesis, writing

$$w_n = \sum_{k=n}^{\infty} a_{k0}\, a_k$$

we obtain

$$a_n = \frac{w_n - w_{n+1}}{a_{n0}}. \qquad \ldots(9)$$

Substituting this value of a_n in (8), we have

$$\varphi_N = \sum_{r=N+1}^{\infty} \frac{w_r - w_{r+1}}{a_{r0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{rk}. \qquad \ldots(10)$$

But for $M > N$

$$\sum_{r=N+1}^{M} \frac{w_r - w_{r+1}}{a_{r0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{rk}$$

(equation continued on p. 1127)

$$= \sum_{r=N+1}^{M} w_r \sum_{k=n}^{N} a_{kn}^{-1} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right)$$

$$+ \frac{w_{N+1}}{a_{N0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{Nk} - \frac{w_{M+1}}{a_{M0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk}. \qquad \ldots(11)$$

Since w_k is the remainder of a convergent series, it follows that $w_k = o(1)$ as $k \to \infty$. Hence by Lemma 2, for each fixed n and N

$$\frac{w_{M+1}}{a_{M0}} \sum_{k=n}^{N} a_{kn}^{-1} a_{Mk} = o(1)\, 0\left(\frac{1}{a_{n0}} \right) = o(1)$$

as $M \to \infty$. Also by identity (6), as $N > n$,

$$\sum_{k=n}^{N} a_{kn}^{-1} a_{Nk} = 0.$$

Hence it follows from (10) and (11) that

$$\varphi_N = \sum_{r=N+1}^{\infty} w_r \sum_{k=n}^{N} a_{kn}^{-1} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right). \qquad \ldots(12)$$

Hence in order that $\varphi_N = o(1)$ as $N \to \infty$ for every (w_r) such that $w_r = o(1)$ as $r \to \infty$, it is necessary and sufficient that

$$\psi_N = \sum_{r=N+1}^{\infty} \left| \sum_{k=n}^{N} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right) a_{kn}^{-1} \right| = O(1) \qquad \ldots(13)$$

as $N \to \infty$, for fixed n.

By hypothesis (3) and Lemma 1 (a), the first term in the inner sigma in ψ_N is negative and all other terms for $k > n$ are non-negative. Hence the modulus of the inner sigma in (13) is not greater than

$$- \left(\frac{a_{rn}}{a_{r0}} - \frac{a_{r-1,n}}{a_{r-1,0}} \right) a_{nn}^{-1} + \sum_{r=n+1}^{N} \left(\frac{a_{rk}}{a_{r0}} - \frac{a_{r-1,k}}{a_{r-1,0}} \right) a_{kn}^{-1} \qquad \ldots(14)$$

and hence, it follows from (14) that

$$\psi_N \leqslant -\sum_{r=N+1}^{\infty}\left(\frac{a_{rn}}{a_{r0}}-\frac{a_{r-1,k}}{a_{r-1,0}}\right)a_{nn}^{-1}$$

$$+\sum_{r=N+1}^{\infty}\sum_{k=n+1}^{N}\left(\frac{a_{rk}}{a_{r0}}-\frac{a_{r-1,k}}{a_{r-1,0}}\right)a_{kn}^{-1}. \qquad \ldots(15)$$

By hypothesis (3), the first expression in (15) is

$$a_{nn}^{-1}\left(\frac{a_{N,n}}{a_{N,0}}-\lim_{m\to\infty}\frac{a_{mn}}{a_{m0}}\right)$$

$$\leqslant a_{nn}^{-1}\frac{a_{nn}}{a_{n0}} = \text{constant}.$$

Since the terms in the second expression in (15) are non-negative, we may invert the order of summation and the double sum becomes

$$\sum_{k=n+1}^{N} a_{kn}^{-1}\sum_{r=N+1}^{\infty}\left(\frac{a_{rk}}{a_{r0}}-\frac{a_{r-1,k}}{a_{r-1,0}}\right)$$

$$=\sum_{k=n+1}^{N} a_{kn}^{-1}\left(\lim_{n\to\infty}\frac{a_{rk}}{a_{r0}}-\frac{a_{Nk}}{a_{N0}}\right)$$

$$\leqslant -\frac{1}{a_{N0}}\sum_{k=n+1}^{N} a_{kn}^{-1}a_{Nk}\left(\text{since } a_{kn}^{-1}\leqslant 0 \text{ for } k>n\right)$$

$$=\frac{a_{N_n}\,a_{nn}^{-1}}{a_{N0}}$$

and this is bounded for fixed n.

This proves (13) and completes the proof of the theorem.

4. Proof of Theorem 2

Since $b_n = o(1)$, it follows from Lemma 1 (b) (ii) that (5) converges for each n; hence a_n is defined by (5).

Now by identity (6)

$$\sum_{\nu=n}^{N} a_{\nu n}\, a_{\nu} = \sum_{\nu=n}^{N} a_{\nu n} \sum_{k=\nu}^{\infty} a_{k\nu}^{-1}\, b_k$$

$$= \sum_{\nu=n}^{N} a_{\nu n} \left(\sum_{k=\nu}^{N} + \sum_{k=N+1}^{\infty} \right) a_{k\nu}^{-1}\, b_k$$

$$= b_n + \sum_{k=N+1}^{\infty} b_k \sum_{\nu=n}^{N} a_{\nu n}\, a_{k\nu}^{-1} .$$

Hence

$$\lim_{N\to\infty} \sum_{\nu=n}^{N} a_{\nu n}\, a_{\nu} = b_n$$

if and only, if, for fixed n,

$$\lim_{N\to\infty} \sum_{k=N+1}^{\infty} b_k \sum_{\nu=n}^{N} a_{\nu n}\ a_{k\nu}^{-1} = 0. \qquad \ldots(16)$$

Since $b_k = o\,(1)$, to prove (16) it is enough to prove that, as $N \to \infty$, for fixed n,

$$\alpha_N = \sum_{k=N+1}^{\infty} \left| \sum_{\nu=n}^{N} a_{\nu n}\ a_{k\nu}^{-1} \right| = O\,(1). \qquad \ldots(17)$$

But since by Lemma 1 (a), $a_{k\nu}^{-1} \leqslant 0\ (k > \nu)$, all the terms in (17) have the same sign and so we may omit the modulus sign. Thus

$$\alpha_N = -\sum_{k=N+1}^{\infty} \sum_{\nu=n}^{N} a_{\nu n}\ a_{\nu k}^{-1}$$

$$= \sum_{\nu=n}^{N} a_{\nu n} \sum_{k=N+1}^{\infty} \left(-\, a_{k\nu}^{-1} \right)$$

$$\leqslant \sum_{\nu=n}^{N} a_{\nu n} \sum_{k=\nu}^{N} a_{k\nu}^{-1} = 1$$

by (iii) and (iv) of Lemma 1 (b).

This proves that (1) holds and the uniquenses of 'a' follows from Theorem 1.

This completes the proof of Theorem 2.

Remark : Theorem 1 is unchanged if we multiply each row of A by a non-zero constant and each column by a non-zero constant; in other words, we replace (a_{nk}) in Theorem 1 by $(\lambda_n \mu_k a_{nk})$ where (λ_n) and (μ_n) are any sequences of non-zero constants.

For suppose that A satisfies the conditions of Theorem 1. Applying that theorem with a_n replaced by $\lambda_n a_n$ and b_n by b_n/μ_n we see that if

$$b_n = \mu_n \sum_{k=n}^{\infty} a_{kn} \lambda_k a_k$$

converges for all n, then

$$a_n = \lambda_n^{-1} \sum_{k=n}^{\infty} a_{kn}^{-1} \lambda_k^{-1} b_k.$$

Thus Theorem 1 applies rather more generally, to any matrix of the form $(\lambda_n \mu_k a_{nk})$ where (a_{nk}) satisfies the conditions in the form stated.

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