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GENERALIZED STIELTJES TRANSFORM OF BANACH SPACE VALUED DISTRIBUTIONS

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In this paper an extension of a generalized Stieltjes transform

$$F(x) = \frac{\Gamma\, b\, \Gamma(\beta+1)}{\Gamma(\alpha+b)} \frac{1}{x} \int_0^\infty \left(\frac{t}{x}\right)^\beta F\left(b, \beta+1; b+\alpha; -\frac{t}{x}\right) f(t)\, dt;$$

$$b = \beta + \eta + 1, \beta \geqslant 0, \eta > 0$$

to Banach space valued distributions (generalized functions) is provided. An inversion formula for the above transform is also proved in distributional (Banach space valued) sense.

1. INTRODUCTION

The theory of distributions has been worked out in great depth by L. Schwartz. Sebastiaoe Silva[1] also has a theory for vector valued distributions. Zemanian[2,3] has presented the theory of Banach space valued distributions. He has discussed the Laplace transform of Banach space valued distributions. Further he has used these concepts for applications in system theory and signals.

The author[4,5] extended the generalized Stieltjes transform, namely

$$F(x) = \frac{\Gamma(\beta+\eta+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+\eta+1)} \frac{1}{x} \cdot \int_0^\infty \left(\frac{t}{x}\right)^\beta F\left(\beta+\eta+1, \beta+1; \alpha+\beta+\eta+1; -\frac{t}{x}\right) f(t)\, dt \qquad \ldots(1.1)$$

to distributions. The complex inversion formula for the transform (1.1) namely

$$f(t) = \frac{1}{2\pi i} \lim_{w\to\infty} \int_{c-wi}^{c+wi} \frac{\Gamma(d+s-\beta-1)}{\Gamma(b+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s}\, t^{-s} M(s)\, ds$$

where

$$M(s) = \int_0^\infty x^{s-1} F(x)\, dx$$

$$b = \beta + \eta + 1,\ d = b + \alpha,\ s = c + iw,\ \beta \geqslant 0,\ \eta > 0$$

was also extended to distributions by the author[4].

Motivated from the work of Zemanian[2,3] and Tiwari [4,5,6] we study generalized Stieltjes transform for Banach space valued distributions, in this paper.

The notation and terminology of this paper will follow that of Zemanian[2,3,7] and Tiwari[4,5,6] when U and V are two topological vector space, the symbol $[U; V]$ denotes the linear space of all continuous linear mappings of U into V. A and B will usually denote Banach spaces. Throughout the sequel we use the following notation :

$\mathbf{R}$ = set of all real numbers

$\mathbb{C}$ = set of all complex numbers

I = $(0, \infty)$.

$D(A)$ = space of smooth A-valued test functions with compact support defined on I.

2. The Spaces $S_a(A)$ and $[S_a(A); B]$

Definition 2.1—The space $S_a(A)$ is defined as the linear space of all smooth functions $\varphi(t)$ from I into A such that

$$\rho_{a,l}(\varphi) = \max_{0 \leqslant k \leqslant l} \sup_{t \in I} \| (1 + t)^a\, t^k\, \varphi^k(t) \|_A < \infty,\ l = 0, 1, 2, \ldots$$

The topology of $S_a(A)$ is generated by sequence of seminorms $\{\rho_{a,l}\}_{l=0}^\infty$. For $A = \mathbb{C}$ we write $S_a(\mathbb{C}) = S_a$. It is easy to see that $D(A) \subset S_a(A)$. If $\{a_v\}_{v=1}^\infty$ be a monotonic sequence of real numbers such that $a_v \to w+$, where w is a real number or $-\infty$, the space $S(w; A)$ is defined as

$$S(w; A) = \bigcup_{v=1}^\infty S_{a_v}(A).$$

For a Banach space B, any $f \in [S_a(A); B]$ will be called a Banach space valued distribution. When $A = B = \mathbb{C}$, f becomes a scalar distribution. The simple or weaker topology for $[S_a(A), B]$ is generated by the collection of seminorms $\{\gamma_\varphi\}_\varphi$, where φ traverses $S_a(A)$ and

$$\gamma_\varphi(f) = \| < f, \varphi > \|_B.$$

We now come to the definition of the space $[S_a(A); B]$.

If $\varphi \in S_a$, $a \in A$, then $\varphi a = a\varphi$ is the function from I into A that assigns to each $t \in I$ the value $\varphi(t)\, a$. Clearly $\varphi\, a \in S_a(A)$. We denote by $S_a \otimes A$ the subspace of $S_a(A)$ consisting of elements of the form $\varphi\, a$. If $g \in [S_a; \mathbb{C}]$ and $a \in A$, $ga = ag$ is defined by

$$<ga, \varphi> = <g, \varphi>\, a, \varphi \in S_a.$$

Clearly $ga \in [S_a; A]$. $[S_a; \mathbb{C}] \otimes A$ denotes the subspace of $[S_a; A]$ consisting of all linear combinations of elements of the form ga. Similarly when $g \in [S_a; [A; B]]$ and $a \in A$, we define $ga = ag \in [S_a; B]$. Every $g \in [S_a(A); B]$ uniquely defines an $f \in [S_a; [A; B]]$ through the equation

$$<f, \varphi>\, a = <g, \varphi a>,\ \varphi \in S_a, a \in A.$$

See Zemanian[2], p. 105.

3. Generalized Stieltjes Transform

It is proved in Tiwari[5] that

$$\frac{\Gamma b\ \Gamma(\beta+1)}{\Gamma d}\ \frac{1}{x}\left(\frac{t}{x}\right)^{\beta}$$

$$F\left(b, \beta+1; d; -\frac{t}{x}\right) \in S_a \text{ for } a \leqslant 1$$

where

$$b = \beta + \eta + 1,\ d = b + \alpha.$$

Now let $f \in [S_a, A]$, we define the generalized Stieltjes transform $F(x)$ of f by

$$F(x) = <f(t), \frac{\Gamma b\ \Gamma(\beta+1)}{\Gamma d}\ \frac{1}{x}\ \left(\frac{t}{x}\right)^{\beta}\ F\left(b, \beta+1; d; -\frac{t}{x}\right)>. \qquad \ldots(3.1)$$

The left hand side $F(x)$ is an A-valued function.

Similarly if $y \in [S_a(A); B]$. The generalized Stieltjes transform $Y(x)$ of y is defined as the generalized Stieltjes transform of f_y :

$$Y(x) = <f_y(t), \frac{\Gamma b\ \Gamma(\beta+1)}{\Gamma d}\ \frac{1}{x}\ \left(\frac{t}{x}\right)^{\beta}\ F\left(b, \beta+1; d; -\frac{t}{x}\right)> \qquad \ldots(3.2)$$

where

$$f_y \in [S_a; [A; B]].$$

The above definition is meaningful because corresponding to y we have the unique f_y as discussed in previous section. Note that $Y(x)$ is an $[A; B]$ valued function.

4. An Inversion Theorem

We need the following there lemmas for the proof of inversion theorem.

Lemma 4.1—Let $y \in [S_a(A); B]$, f_y be the corresponding member in $[S_a; [A; B]]$ and

$$\theta(x, \tau) = \frac{\Gamma b\, \Gamma(\beta+1)}{\Gamma d} \frac{1}{x} \left(\frac{\tau}{x}\right)^{\beta} F\left(b, \beta+1; d; -\frac{\tau}{x}\right).$$

Then

$$\int_0^{\infty} x^{-s} \langle y(\tau), \theta(x, \tau) \rangle\, dx$$

$$= \langle y(\tau), \int_0^{\infty} \theta(x, \tau)\, x^{-s}\, dx \rangle.$$

Lemma 4.2—Let $y \in [S_a(A), B]$, $\varphi \in D(A)$ and

$$P(s) = \int_0^{\infty} \varphi(t)\, t^{-s}\, dt.$$

Then, for any two fixed real numbers r and c such that $0 < r < \infty$, $s = c + iw$

$$\int_{-r}^{r} \langle f_y(\tau), \tau^{s-1} \rangle\, P(s)\, dw$$

$$= \langle y(\tau), \int_{-r}^{r} \tau^{s-1} P(s)\, dw \rangle.$$

Lemma 4.3—If $\varphi \in D(A)$ and a, c, r be real numbers with $a < c$. Then

$$\frac{1}{\pi} \int_0^{\infty} \frac{\varphi(v)}{u \log\left(\frac{u}{v}\right)} \left(\frac{u}{v}\right)^{c} \sin\left(r \log \frac{u}{v}\right) dv$$

converges in $S_a(A)$ to $\varphi(u)$ as $r \to \infty$.

Except for some obviuos changes proofs of the above lemmas are respectively similar to those of Lemmas 4.1, 4.2 and 4.3 in Tiwari[4]. We now come to inversion theorem.

Theorem 4.1—If $y \in [D(A); B]$ and $Y(x)$ be the generalized Stieltjes transform of Banach space valued distribution y, then in the sense of weak convergence in $[D(A); B]$.

$$\lim_{r\to\infty} < \frac{1}{2\pi i}\int_{c-ir}^{c+ir} \frac{\Gamma(d+s-\beta-1)}{\Gamma(b+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma(s)} M(s)\, t^{-s}\, ds, \varphi(t)>$$

$$= <y, \varphi>$$

where

$$M = \int_0^\infty x^{s-1}\,\gamma(x)\,dx.$$

Proof : On substituting

$$\frac{\Gamma(d+s-\beta-1)}{\Gamma(b+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s} = Q$$

and

$$s = c + iw$$

we have

$$< \frac{1}{2\pi i}\int_{c-ir}^{c+ir} \frac{\Gamma(d+s-\beta)}{\Gamma(b+s-\beta-1)\,\Gamma(\beta+1-s)\,\Gamma s} M(s)\, t^{-s}\, ds, \varphi(t)>$$

$$= < \frac{1}{2\pi}\int_{-r}^{r} Q M(s)\, t^{-s}, \varphi(t)>. \qquad ...(4.1)$$

The integral on w is an $[A; B]$ valued continuous function of t. By the continuity of the B-valued function $Y(x)\,\varphi(t)$ on the $w - t$ plane and using change of order of integration (4.1)

$$= \frac{1}{2\pi}\int_{-r}^{r} Q\, M(s) \int_0^\infty t^{-s}\,\varphi(t)\,dt\,dw$$

$$= \frac{1}{2\pi}\int_{-r}^{r} Q \int_0^\infty x^{s-1}\,\gamma(x)\,dx \int_0^\infty t^{-s}\,\varphi(t)\,dt\,dw$$

$$= \frac{1}{2\pi}\int_{-r}^{r} Q \int_0^\infty x^{s-1} <y(\tau), \frac{\Gamma b\,\Gamma(\beta+1)}{\Gamma d}\frac{1}{x}\left(\frac{\tau}{x}\right)^{\beta}.$$

(equation continued on p. 1136)

$$F\left(b, \beta+1; d; -\frac{\tau}{x}\right)> dx \int_0^\infty t^{-s}\varphi(t)\,dt\,dw \qquad \ldots(4.2)$$

$$= \frac{1}{2\pi}\int_{-r}^{r} Q<y(\tau), \int_0^\infty x^{s-1}\frac{\Gamma b\,\Gamma(\beta+1)}{\Gamma d}\frac{1}{x}\left(\frac{\tau}{x}\right)^{\beta}$$

$$\times F\left(b, \beta+1; d; -\frac{\tau}{x}\right)dx> \int_0^\infty t^{-s}\varphi(t)\,dt\,dw \qquad \ldots(4.3)$$

by Lemma (4.1).

Using the result of Tiwari[4] (p. 1052), viz.

$$Q\int_0^\infty x^{s-1}\int_0^\infty \frac{\Gamma b\,\Gamma(\beta+1)}{\Gamma d}\frac{1}{x}\left(\frac{\tau}{x}\right)^{\beta} F\left(b, \beta+1; d; -\frac{\tau}{x}\right)$$

$$\times y(\tau)\,d\tau\,dx$$

$$= \int_0^\infty \tau^{s-1}y(\tau)\,d\tau$$

we have

$$(4.3) = \frac{1}{2\pi}\int_{-r}^{r} <y(\tau), \tau^{s-1}> \int_3^\infty t^{-s}\varphi(t)\,dt\,dw$$

$$= <y(\tau), \frac{1}{2\pi}\int_{-r}^{r}\tau^{s-1}\int_0^\infty t^{-s}\varphi(t)\,dt\,dw> \qquad \ldots(4.4)$$

by Lemma (4.2).

The proof now can be completed as in Tiwari[4] (p. 1053).

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