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ON SINGLETONNESS OF UNIQUELY REMOTAL SETS

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It is shown that a bounded subset G of a finite dimensional Hilbert space H, whose closure, $\bar{G}$, admits a unique farthest point to its Chebyshev center is necessarily a singleton. The same result also holds when H is infinite dimensional and G is nearly compact.

For a non-empty bounded subset G of a normed linear space X, the farthest distance function from G is defined by

$$t(x) = \sup_{a \in G} \|x - a\|, (x \in X).$$

The subset G is said to be uniquely remotal (or to have the unique farthest point property) in X if for each $x \in X$ there exists a unique element $q(x) \in G$ such that $t(x) = \|x - q(x)\|$. The following result is known for finite dimensional Hilbert spaces[2].

Proposition—Every uniquely remotal subset G of a finite dimensional Hilbert space is necessarily a singleton.

We also know[3] that the unique farthest point property of a uniquely remotal subset G of a strictly convex normed linear space is inherited by $\bar{G}$, the norm closure of G. Knowing this, and the fact that Hilbert spaces are strictly convex, we can (equivalently) restate the above proposition as follows :

Theorem —Let G be a subset of a finite dimensional Hilbert space H, such that $\bar{G}$ admits a unique farthest point to each $x \in H$. Then G is necessarily a singleton.

We now strengthen the above Theorem, as the following main result of this note. Recall that every bounded subset G of a Hilbert space has a unique Chebyshev center c (see Astaneh[1], p. 1312); i. e., there exists a unique $c \in H$ such that $t(c) = \inf_{x \in H} t(x)$. The non-negative real number $t(c)$ is called the Chebyshev radius of G and is denoted by $r(G)$.

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Theorem 1—Let G be a bounded subset of a finite dimensional (real or complex) Hilbert space H. If $\bar{G}$ admits a unique farthest point to the Chebyshev center c of G, then G is necessarily a singleton.

PROOF : Suppose G is not a singleton. Without any loss of generality we may assume that $r(G) = 1$ and $c = 0$ is the Chebyshev center of G. Let x_0 denote the unique farthest point for $c = 0$ in $\bar{G}$. Then x_0 is the only point in $\bar{G} \cap U(H)$ with $\|x_0\| = 1$. (Here $U(H)$ is the closed unit ball of H). Consider the sequence $x_n = x_0/n$, $(n > 2)$ in the open line segment $]0, x_0[$, and note that for each n

$$1 < t(x_n)^2 = \sup_{a \in G} \|x_n - a\|^2.$$

(The strict inequality on the left is due to the uniqueness of the center 0 of G, which does not lie on $]0, x_0[$). Therefore, for each n we can choose $a_n \in G$ such that

$$1 < \|x_n - a_n\|^2 \leqslant t(x_n)^2. \qquad \ldots(1)$$

Clearly for each n we have $\|a_n\| < 1$, and hence by (1),

$$\begin{aligned} 1 > \|a_n\|^2 &= \|x_n - a_n\|^2 - \|x_n\|^2 + 2\,\mathrm{Re} \langle x_n, a_n \rangle \\ &= \|x_n - a_n\|^2 - \frac{1}{n^2} + \left(\frac{2}{n}\right) \mathrm{Re} \langle x_0, a_n \rangle \\ &> 1 - \frac{1}{n^2} + \left(\frac{2}{n}\right) \mathrm{Re} \langle x_0, a_n \rangle. \end{aligned}$$

Therefore, for each n the following inequality holds

$$2\mathrm{Re} \langle x_0, a_n \rangle < \frac{1}{n}. \qquad \ldots(2)$$

Considering (1) and (2), for each n we have

$$\begin{aligned} \|x_0 - a_n\|^2 &= \|x_0 - x_n\|^2 + \|x_n - a_n\|^2 + 2\,\mathrm{Re} \langle x_0 - x_n, x_n - a_n \rangle \\ &= \left(1 - \frac{1}{n}\right)^2 + \|x_n - a_n\|^2 + 2\left(1 - \frac{1}{n}\right) \\ &\qquad \mathrm{Re} \langle x_0, x_n - a_n \rangle \\ &= \left(1 - \frac{1}{n}\right)^2 + \|x_n - a_n\|^2 + 2\left(1 - \frac{1}{n}\right)\frac{1}{n} \\ &\qquad - 2\left(1 - \frac{1}{n}\right) \mathrm{Re} \langle x_0, a_n \rangle \\ &> \left(1 - \frac{1}{n}\right)^2 + 1 + 2\left(1 - \frac{1}{n}\right)\frac{1}{n} - \left(1 - \frac{1}{n}\right) \\ &\qquad \frac{1}{n} = 2 - \frac{1}{n}. \end{aligned}$$

Therefore, for each $n > 2$ we get

$$\|x_0 - a_n\| > \sqrt{\frac{3}{2}}. \qquad \ldots(3)$$

On the other hand since the farthest distance function $x \to t(x)$ is continuous, and since $x_n \to 0$ we have $\lim_n (x_n) = t(0) = 1$. From this and (1) it follows that $\|x_n - a_n\| \to 1$. Next from $\|x_n - a_n\| \to 1$ and

$$\|x_n - a_n\| - \frac{1}{n} = \|x_n - a_n\| - \|x_n\| \leqslant \|a_n\| < 1$$

we deduce that $\lim_n \|a_n\| = 1$.

This means that $\{a_n\}$ is a maximizing sequence for 0 in G. We now observe that, H being finite dimensional, $\{a_n\}$ has a cluster point a_0 which of course lies in $\bar{G}$. Since $\|a_0\| = 1$ and $\|x_0 - a_0\| \geqslant \sqrt{\frac{3}{2}}$ (by (3)), it follows that 0 has another farthest point a_0 in $\bar{G}$ distinct from x_0. This contradiction completes the proof of the theorem.

We observe that the finite dimensionality assumption on H in the above proof is not used until the last paragraph, where we require a cluster point for the maximizing sequence $\{a_n\}$ for 0 in G. This observation enables us to deduce the following theorem in the infinite dimensional case, as well.

Theorem 2—Let G be a bounded subset of an infinite dimensional (real or complex) Hilbert space H. Let $\bar{G}$ admit a unique farthest point to Chebyshev center c of G, and assume that every maximizing sequence for c in G has a cluster point (i. e. G is nearly compact). Then G is necessarily a singleton.

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