Indian J. pure appl. Math., 17 (9) : 1140–1144 September 1986

ON THE COEFFICIENTS OF POWERS OF A CLASS OF BAZILEVIC FUNCTIONS

M. JAHANGIRI

Department of Mathematics, University of York, York, Great Britain

(*Received* 23 *November* 1984; *after revision* 12 *May* 1986)

Sharp estimates for coefficients of powers of a class of Bazilevic functions are given and the smallest subclass, yet known, of univalent functions, for which the powers of the Koebe function are not extremal, is introduced.

INTRODUCTION

Suppose that $0 < \alpha < \infty$. For functions $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ analytic and univalent in the open unit disk $U = \{z: |z| < 1\}$ and for the Koebe function $k(z) = z(1 - z)^{-2}$ we write

$$\left\{\frac{f(z)}{z}\right\}^{1/\alpha} = \sum_{n=0}^{\infty} a_n(\alpha) z^n \text{ and } \left\{\frac{k(z)}{z}\right\}^{1/\alpha} = \sum_{n=0}^{\infty} b_n(\alpha) z^n \qquad \ldots(1)$$

so that

$$a_1(\alpha) = \frac{1}{\alpha} a_2,\ a_2(\alpha) = \frac{1}{\alpha}\left[a_3 - \frac{\alpha - 1}{2\alpha} a_2^2\right] \text{ and}$$

$$b_n(\alpha) = \frac{2(2+\alpha)(2+2\alpha)\ldots(2+(n-1)\alpha)}{(n!)\,\alpha^n}. \qquad \ldots(2)$$

We consider the inequality

$$|a_n(\alpha)| \leqslant b_n(\alpha). \qquad \ldots(3)$$

If $\alpha = 1$, this is the Bieberbach conjecture[3] which has recently been proved by De Branges[4] and for $\alpha = 2$ it is the Littlewood-Paley[10] conjecture which was disproved by Fekete-Szegö[5].

In fact (3) is true for univalent functions if $\alpha \leqslant 1$ and is false for $\alpha > 1$ and $n = 2$. (See Hayman and Hummel[6]).

We may ask whether (3) is still true for subclasses of univalent functions. In this paper we introduce the smallest class, yet known, of univalent functions for which the

Present Address : Department of Mathematics, University of Kentucky, Lexington, KY 40506-0027 U. S. A.

inequality (3) is false. Much of the work of this paper is drawn from the author's D. Phil. thesis[7].

For $\beta > 0$ let $B(\beta)$, called Bazilevic of type β, denote the class of functions $f(z)$ analytic in U satisfying the differential equation

$$zf'(z)(\{f(z)\}^{\beta-1} = (\{g(z)\}^{\beta} p(z) \qquad \ldots(4)$$

where g is a normalized starlike function so that

$$\frac{zg'(z)}{g(z)} = q(z). \qquad \ldots(5)$$

Here $p(z)$ and $q(z)$ are functions of positive real part with $p(0) = q(0) = 1$. It has been proved by many authors including Bazilevic[2] that the functions in $B(\beta)$ are univalent. (See also Sheil-Small[11,12]).

Note that $B(1)$ is the class of normalised close-to-convex functions defined by Kaplan[8].

Theorem 1—Suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belongs to $B(1)$. Then we have the sharp bounds

$$|a_2(\alpha)| \leqslant \begin{cases} \dfrac{2+\alpha}{\alpha^2} & \text{if } 0 < \alpha \leqslant 3 \\ \dfrac{11\alpha-3}{9\alpha(\alpha-1)} & \text{if } 3 < \alpha < \infty. \end{cases}$$

Corollary 1—For $3 < \alpha < \infty$, Theorem 1 shows that the inequality (3) does not hold for powers of functions in $B(1)$.

Baernstein[1] proved that if $f(z) = z + \ldots$ is univalent, then for $h_4(z) = \{f(z^4)\}^{1/4} = z + \sum_{n=1}^{\infty} b_{4n+1} z^{4n+1}$, $|b_{4n+1}| \leqslant Kn^{-1/2}$ where K is an absolute constant. Keogh and Miller[9] obtained tke estimate $|b_{mn+1}| \leqslant \left[\dfrac{n-1+2/m}{n}\right]$ for $h_m(z) = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1}$ if $h_m \in B(\beta)$ and $0 < \beta \leqslant 1$. Little is known about the coefficients of $h_m(z)$ in $B(m)$ when m is an integer and $m \geqslant 2$. We prove that

Theorem 2—Suppose that $h_m(z) = z + \sum_{n=1}^{\infty} b_{mn+1} z^{mn+1}$ belongs to $B(m)$ where m is a positive integer. Then we have the sharp bounds

$$|b_{2m+1}| \leqslant \begin{cases} \dfrac{2+m}{m^2} & \text{if } m \leqslant 3 \\ \dfrac{11m-3}{9m(m-1)} & \text{if } m \geqslant 4. \end{cases}$$

Corollary 2—For $m = 2$, Theorem 2 shows that Fekete-Szegö's[5] disproof of the Littlewood-Paley[10] conjecture does not apply to the square-root transform of close-to-convex functions.

Theorem 3—If m is an integer, $m > 3$, consider the meromorphic univalent function $g(\xi) = \{f(z^m)\}^{-1/m} = \xi + c_{m-1}\xi^{1-m} + c_{2m-1}\xi^{1-2m} + \ldots$

where $f(z) \in B(1)$, $z = 1/\xi$ and $|\xi| > 1$. Then we have the sharp bound

$$\left|c_{2m-1}\right| \leqslant \frac{11m+3}{9m(m+1)}.$$

If $m = 1$, the classical estimate $|c_1| \leqslant 1$ is sharp. The case $m = 2$ is more difficult and we omit it.

The estimate given in Theorem 3 corresponds to the estimate $|c_{2m-1}| \leqslant m^{-1}\{1 + 2e^{-2(m+1)/(m-1)}\}$ obtained by Fekete-Szegö[5] for the whole class of m-fold symmetric meromorphic univalent functions.

To prove our theorems we shall need the following lemma.

Lemma 1—Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be such that $\operatorname{Re} p(z) > 0$ in $|z| < 1$. Then

$\left|p_2 + \alpha p_1^2\right| \leqslant 2 + \alpha |p_1|^2$ if $\alpha \geqslant -1/2$.

PROOF : Write $p(z) = \{1 + \omega(z)\}/\{1 - \omega(z)\}$ where $\omega(z) = w_1 z + w_2 z^2 + \ldots$ and $|\omega(z)| \leqslant |z| < 1$. Equate the coefficients of z and z^2 to obtain $p_1 = 2w_1$ and $p_2 = 2\left(w_2 + w_1^2\right)$. By applying the fact that $|w_2| \leqslant 1 - |w_1|^2$ we obtain $|p_2 - 1/2\, p_1^2| \leqslant 2 - 1/2\,|p_1|^2$. Now for $\alpha \geqslant -1/2$ add $(\alpha + \frac{1}{2})|p_1|^2$ to the both sides of the latter inequality to obtain the lemma.

Proof of Theorem 1—We see that the coefficients of functions $f(z)$ in $B(1)$ can be expressed by means of (4) and (5) in terms of the coefficients of two positive real part functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ and $q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n$ and so can the coefficients $a_n(\alpha)$ of $\{z^{-1} f(z)\}^{1/\alpha}$ defined by (1).

Therefore we obtain $a_1(\alpha) = (p_1 + q_1)/(2\alpha)$ and

$$a_2(\alpha) = \frac{1}{\alpha}\left\{\frac{1}{6}q_2 + \frac{1}{3}p_2 + \frac{\alpha+3}{24\alpha}q_1^2 - \frac{\alpha-1}{8\alpha}p_1^2 + \frac{\alpha+3}{12\alpha}p_1 q_1\right\}. \quad \ldots(6)$$

It is easy to see that $|a_1(\alpha)| \leqslant 2/\alpha$ since $|p_1| \leqslant 2$ and $|q_1| \leqslant 2$.

For $a_2(\alpha)$ we obtain

$$\alpha\left|a_2(\alpha)\right| \leqslant \frac{1}{6}\left|q_2+\frac{\alpha+3}{4\alpha}q_1^2\right|+\frac{1}{3}\left|p_2-\frac{3(\alpha-1)}{8\alpha}p_1^2\right| + \frac{\alpha+3}{12\alpha}\left|p_1 q_1\right|$$

and so by Lemma 1 it follows that

$$\alpha\,|\,a_2(\alpha)\,| \leqslant \frac{1-\alpha}{8\alpha}\,|\,p_1\,|^2+\frac{\alpha+3}{6\alpha}\,|\,p_1\,|+\frac{7\alpha+3}{6\alpha}=R(\,|\,p_1\,|\,).$$

We require the maximum of $R(\,|\,p_1\,|\,)$ when $0 \leqslant |\,p_1\,| \leqslant 2$ and α is fixed and positive. We note that $R(0)=(7\alpha+3)/(6\alpha)$ and $R(2)=(\alpha+2)/\alpha=\alpha b_2(\alpha)$. Thus the maximum of $a_2(\alpha)$ is greater than $b_2(\alpha)$ if and only if $R(x)$ attains its maximum at p_1^0 in the interval (0.2). Now from

$$\frac{d}{d\,|\,p_1\,|}R(\,|\,p_1\,|\,)=\frac{1-\alpha}{4\alpha}\,|\,p_1\,|+\frac{\alpha+3}{6\alpha}=0.$$

We obtain

$$|\,p_1^0\,|=\{2(\alpha+3)\}/\{3(\alpha-1)\} \text{ and so } R(\,|\,p_1^0\,|\,)=(11\alpha-3)/\{9(\alpha-1)\}.$$

We observe that $R(\,|\,p_1^0\,|\,) > \max\{R(0), R(2)\}$ if $3<\alpha<\infty$ and $R(2)>R(0)$ if $0<\alpha\leqslant 3$ since $|\,p_1^0\,|>2$ when $0<\alpha<3$. This yields the bounds in Theorem 1. To show that they are sharp we write $q_1=q_2=p_2=2$ and $p_1=2$ for $0\leqslant\alpha\leqslant 3$, $p_1=\{2(\alpha+3)\}/\{3(\alpha-1)\}$ if $\alpha>3$. This corresponds to $p(z)=(1+\omega)/(1-\omega)$ where $\omega=z(p_1+2z)/(2+p_1z)$ and $q(z)=(1+z)/(1-z)$ and hence $f(z)$ is constructed from (4) and (5).

To prove Theorem 2 we shall need the following well-known lemma. (See also Sheil-Small[11], p. 142).

Lemma 2—Suppose that n is a positive integer. Then $f(z)\in B(\beta)$ if and only if $\{f(z^n)\}^{1/n}\in B(n\beta)$.

Proof of Theorem 2—For $h_m(z)=\{f(z^m)\}^{1/m}\in B(m)$ we obtain by Lemma 2 that $f(z)\in B(1)$. Since for $f(z)\in B(1)$ both functions $\{z^{-1}f(z)\}^{1/m}$ and $\{z^{-m}f(z^m)\}^{1/m}$ have the same coefficients which are defined by (1) for $m=\alpha$, we apply Theorem 1 to $f(z)$ noting that $b_{2m+1}=a_2(m)$. Therefore we deduce Theorem 2.

Proof of Theorem 3—Since $f(z)\in B(1)$, its coefficients can be expressed in terms of the coefficients of two positive real part functions $p(z)=1+p_1z+\ldots$ and

$q(z) = 1 + q_1 z + \ldots$ and so can the coefficients of $g(\xi) = \{f(z^m)\}^{-1/m}$. In fact with the notation of (1) we have $c_{2m-1} = a_2(-m)$. Therefore by applying (6) we obtain

$$-m\,c_{2m-1} = \frac{1}{6}q_2 + \frac{1}{3}p_2 + \frac{m-3}{24m}q_1^2 - \frac{m+1}{8m}p_1^2 + \frac{m-3}{12m}p_1 q_1.$$

Thus if $m \geqslant 3$ we obtain

$$m\left|c_{2m-1}\right| \leqslant \frac{1}{6}|q_2| + \frac{m-3}{24m}|q_1|^2 + \frac{m-3}{12m}|p_1 q_1| + \frac{1}{3}\left|p_2 - \frac{3(m+1)}{8m}p_1^2\right|.$$

Since $1/2 \geqslant 3(m+1)/8m$, by Lemma 1 we deduce that

$$m\left|c_{2m-1}\right| \leqslant -\frac{m+1}{8m}|p_1|^2 + \frac{m-3}{6m}|p_1| + \frac{7m-3}{6m} = R(|P_1|).$$

We observe that $R(0) = (7m-3)/(6m)$ and $R(2) = (m-2)/m$. Also from

$$\frac{d}{d|p_1|}R(|p_1|) = -\frac{m+1}{4m}|p_1| + \frac{m-3}{6m} = 0$$

we obtain $|p_1^0| = (2m-6)/(3(m+1))$. Now it is easy to see that $R(|p_1|)$ has a maximum at $|p_1^0|$ because for $m \geqslant 3$, $R(|p_1^0|) = (11m+3)/\{(9(m+1)\} \geqslant \max\{R(0), R(2)\}$ and so the desired result follows.

References

1. A. Baernstein II, Coefficients of univalent functions with restricted maximum modulus. To appear.
2. I. E. Bazilevic, *Math. Sb. N. S.* **37** (1955), 471–76.
3. L. Bieberbach, Uber die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, *Sitzungsberichte preussische Akademie der Wissenschaften* (1916), 940–55.
4. L. De Branges, *Acta Math.* **145** (1985), 137–52.
5. M. Fekete, and G. Szegö, *J. London Math. Soc.* **8** (1933) 85–89.
6. W. K. Hayman, and J. A. Hummel, Coefficients of powers of univalent functions, Complex Variables, To appear.
7. M. Jahangiri, Ph. D. Thesis, University of York, Great Britain 1985.
8. W. Kaplan, *Mich. Math. J.* **1** (1952), 169–85.
9. F. R. Keogh, and S. S. Miller, *Proc. Am. Math. Soc.* **30** (1971), 492–96.
10. J. E. Littlewood, and R. E. A. C. Paley, *J. London Math. Soc.* **7** (1932), 167–69.
11. T. B. Sheil-Small. *Quart. J. Math.* (2) **23**, (1972) 135–42.
12. T. B. Sheil-Small, *J. Analyse Math.* **43** (1983/84), 1–11.